Regular completions of $\mathbb{Z}^n$-free groups

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$\mathbb{R}$-trees

An $\mathbb{R}$-tree ia a metric space (X, p) (where $p : X \times X \to \mathbb{R}$) which satisfies the properties:

1. (X, p) is geodesic,
2. if two segments of (X, p) intersect in a single point, which is an endpoint of both, then their union is a segment,
3. the intersection of two segments with a common endpoint is also a segment.

$\mathbb{R}$-trees were introduced by J.Tits in 1977.

The theory of $\mathbb{R}$-trees was developed by Morgan and Shalen (1985), Culler and Morgan (1987).

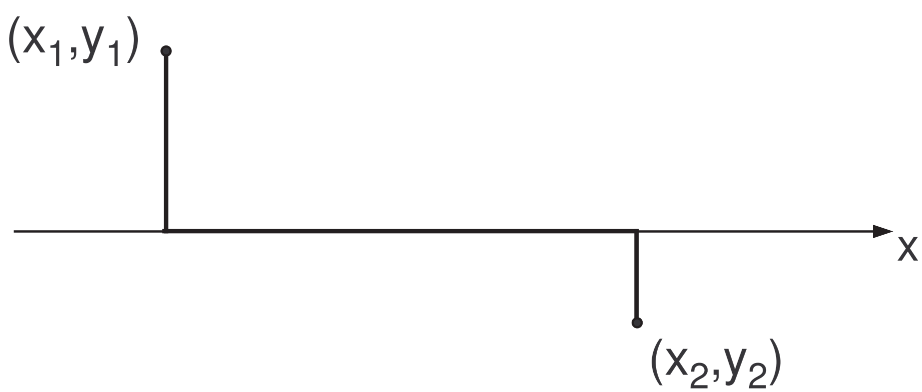
Examples

1. $X = \mathbb{R}$ with usual metric.

2. geometric realization of a simplicial tree.

3. $X = \mathbb{R}^2$ with metric d defined by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1| + |y_2| + |x_1 - x_2| & \text{if } x_1 \neq x_2 \\ |y_1 - y_2| & \text{if } x_1 = x_2 \end{cases}$$



By action of a group G on an $\mathbb{R}$-tree (X, p) we understand an action by isometries.

An action is free if there are no inversions and stabilizers of points are trivial.

$\mathbb{R}$-free groups = groups acting freely on $\mathbb{R}$-trees

Lyndon's conjecture. Every f.g. $\mathbb{R}$-free group is a subgroup of $\mathbb{R} * \cdots * \mathbb{R}$.

Counterexamples to Lyndon's conjecture were given by Alperin and Moss (1985), and Promislow (1985).

Morgan and Shalen (1991) showed that almost all surface groups are $\mathbb{R}$-free.

Rips' Theorem (1991). A f.g. $\mathbb{R}$-free group is a free product of surface groups (with a few exceptions) and free abelian groups of finite rank.

Gaboriau, Levitt, Paulin (1994) gave a complete proof of the Rips' Theorem.

Bestvina and Feighn (1995) gave a proof of the Rips' Theorem using Makanin techniques, and obtained a more general result for stable actions on $\mathbb{R}$-trees.

Remark. Rips' result can't be transfered to infinitely generated groups. Counterexamles were constructed by Dunwoody (1997) and Zastrow (1998).

Ordered abelian groups

Let A be an ordered abelian group (any $a, b \in A$ are comparable and for any $c \in A: \quad a \leq b \;\Rightarrow\; a + c \leq b + c$).

Examples.

1. archimedean case: $A = \mathbb{R}$, $A = \mathbb{Z}$ with usual order.
2. non-archimedean case: $A = \mathbb{Z}^2$ with the right lexicographic order

$$(a, b) < (c, d) \iff b < d \text{ or } b = d \text{ and } a < c.$$

In particular,

$$(0, 1) > (n, 0) \text{ for every } n \in \mathbb{Z}.$$

Λ-free groups

Morgan and Shalen (1985) defined Λ-trees.

A Λ-tree is a Λ-metric space enjoying the properties listed in the definition of $\mathbb{R}$-trees with $\mathbb{R}$ substituted by Λ.

Alperin and Bass (1987) developed the theory of Λ-trees and stated the fundamental problem:

Problem. Find the group theoretic information carried by an action on a Λ-tree.

Motivation: Bass-Serre theory for simplicial trees = description of groups acting on $\mathbb{Z}$-trees (1977).

Some properties of groups acting freely on Λ-trees (Λ-free groups)

1. The class of Λ-free groups is closed under taking subgroups and free products.

2. Λ-free groups are torsion-free.

3. Λ-free groups have the CSA-property (maximal abelian subgroups are malnormal).

4. Commutativity is a transitive relation on the set of non-trivial elements.

5. Any two-generator subgroup of a Λ-free group is either free or free abelian.

Bass (1991) proved that a Λ-free group G, where $\Lambda = \Lambda' \oplus \mathbb{Z}$ with the right lexicographic order and Λ' is an ordered abelian group, is a fundamental group of a graph of groups with several properties. In particular,

1. vertex groups are Λ'-free,
2. edge groups are maximal abelian subgroups in corresponding vertex groups, and all edge groups are embeddable into Λ'.

Since $\mathbb{Z}^n$ with the right lexicographic order can be viewed as $\mathbb{Z}^{n-1} \oplus \mathbb{Z}$ then the structure of $\mathbb{Z}^n$-free groups follows.

Examples of $\mathbb{Z}^n$-free groups:

1. $\mathbb{R}$-free groups,
2. fully residually free groups (limit groups),
3. the group $\langle\, x_1, x_2, x_3 \mid x_1^2\, x_2^2\, x_3^2 = 1 \,\rangle$ is $\mathbb{Z}^2$-free, but it is neither $\mathbb{R}$-free, nor fully residually free.

Martino and Rourke (2005) proved that for $\mathbb{Z}^n$-free groups G_1 and G_2, the amalgamated product $G_1 *_C G_2$ along a cyclic subgroup C maximal in both factors is $\mathbb{Z}^m$-free for some $m \in \mathbb{N}$.

From actions to length functions

Let G be a group acting on a Λ-tree (X, d). Fix a point $x_0 \in X$ and consider a function $l : G \to \Lambda$ defined by

$$l(g) = d(x_0\ ,\ gx_0)$$

l is called a based length function on G with respect to x_0, or a Lyndon length function.

l is free if the underlying action is free.

Example: In a free group F, the function $f \to |f|$ is a free $\mathbb{Z}$-valued (Lyndon) length function.

Abstract length functions (Lyndon, 1963)

Let G be a group. A function $l : G \longrightarrow \Lambda$ is called a (Lyndon) length function on G if

(L1) $\forall\ g \in G:\ \ l(g) \geqslant 0$ and $l(1) = 0$,

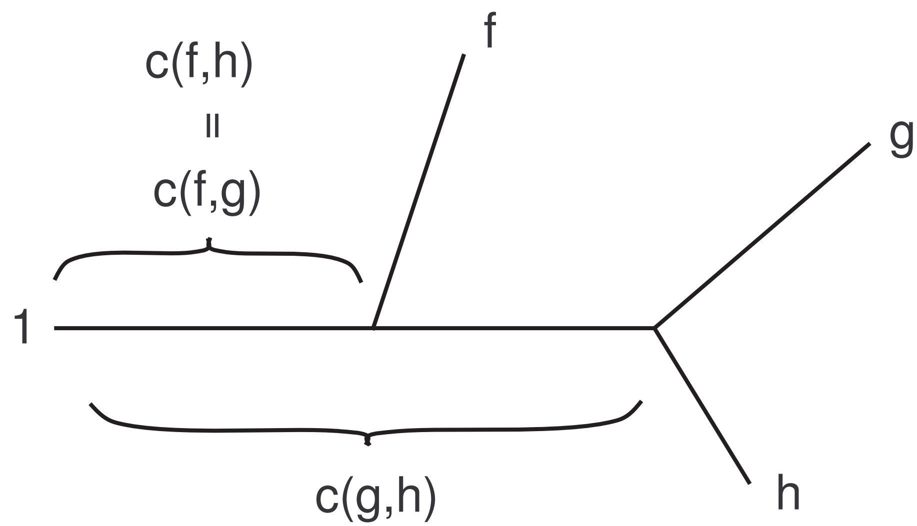
(L2) $\forall\ g \in G:\ \ l(g) = l(g^{-1})$,

(L3) the triple $\{c(g, f), c(g, h), c(f, h)\}$ is isosceles for all $g, f, h \in G$, where

$$c(g, f) = \frac{1}{2}(l(g) + l(f) - l(g^{-1}f)).$$

$\{a, b, c\}$ is isosceles = at least two of a, b, c are equal, and not greater than the third.

Example. In a free group F, the function $f \to |f|$ is a $\mathbb{Z}$-valued length function. For $f, g, h \in F$ we have



A length function $l : G \to \Lambda$ defines a Λ-pseudometric d on G:

$$d(f, g) = l(fg^{-1})$$

$l : G \to \Lambda$ is free if

(L4) for all $g \in G,\ g \neq 1 \to l(g^2) > l(g)$.

Lyndon (1963) proved that a group G has a free length function in $\mathbb{Z}$ if and only if G is free.

Chiswell (1976) established a connection between real-valued length functions and actions on metric spaces, which were proved to be $\mathbb{R}$-trees afterwards by Imrich (1977).

Morgan and Shalen (1985) - generalization of Chiswell's result to Λ-valued length functions for arbitrary Λ.

From length functions to infinite words

Let A be a discretely ordered abelian group with a minimal positive element 1_A and $X = \{x_i \mid i \in I\}$ be a set.

An A-word is a function of the type

$$w : [1_A, \alpha] \to X^{\pm},$$

where $\alpha \geq 0$. The element α is called the length $|w|$ of w.

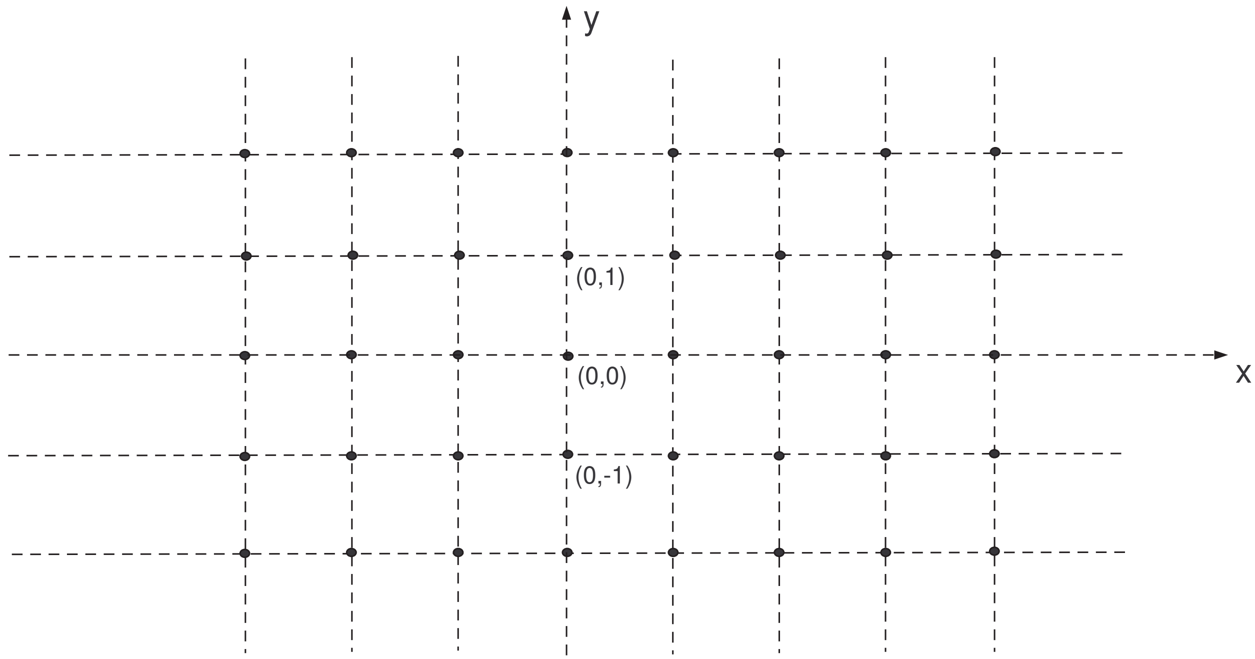
By ε we denote the empty A-word (when $\alpha = 0$).

w is reduced $\iff$ no subwords xx^{-1}, $x^{-1}x$ $(x \in X)$.

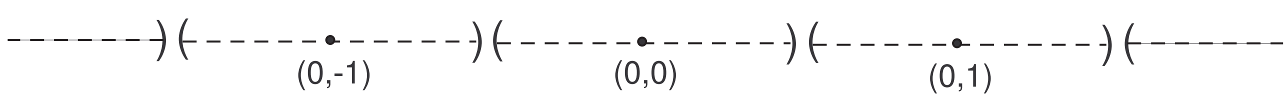
$R(A, X) =$ the set of all reduced A-words.

Example. $A = \mathbb{Z}^2$

$\mathbb{Z}^2$



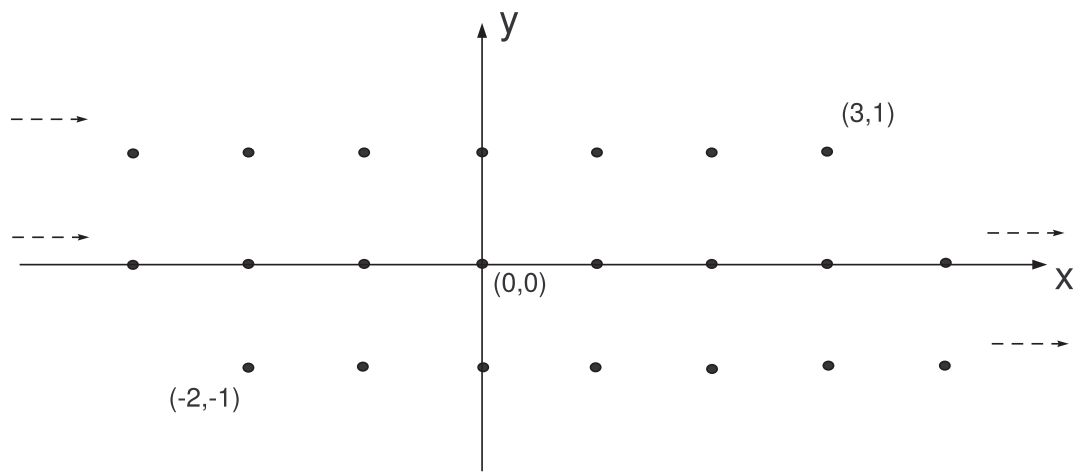
$\mathbb{Z}^2$ with the right lexicographic order

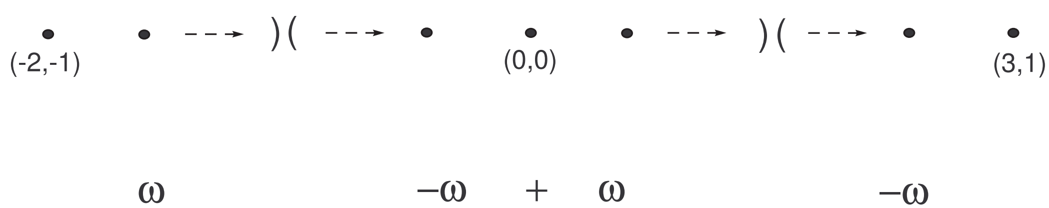


For $\alpha, \beta \in \mathbb{Z}^2$ the closed segment $[\alpha, \beta]$ is defined by

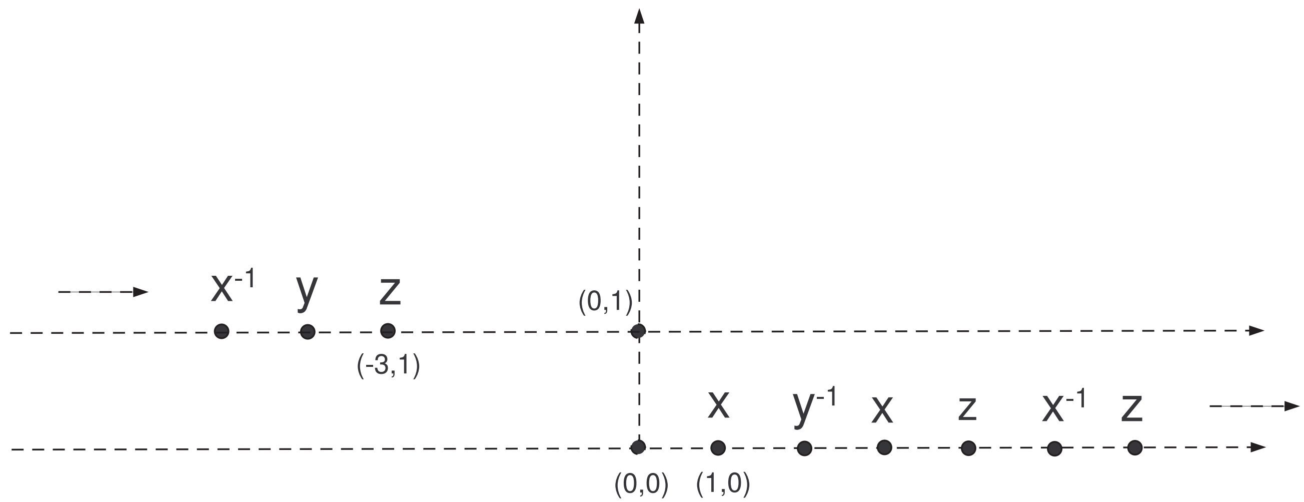
$$[\alpha, \beta] = \{\gamma \in \mathbb{Z}^2 \mid \alpha \leq \gamma \leq \beta \}.$$

Example. $[(-2,-1),(3,1)]$

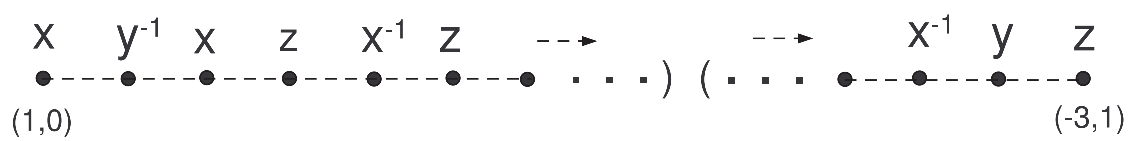




Example. $X = \{x, y, z\},\ A = \mathbb{Z}^2$



In "linear" notation



Concatenation and inversion of A-words can be defined as in a free group.

Multiplication

Let $u, v \in R(A, X)$.

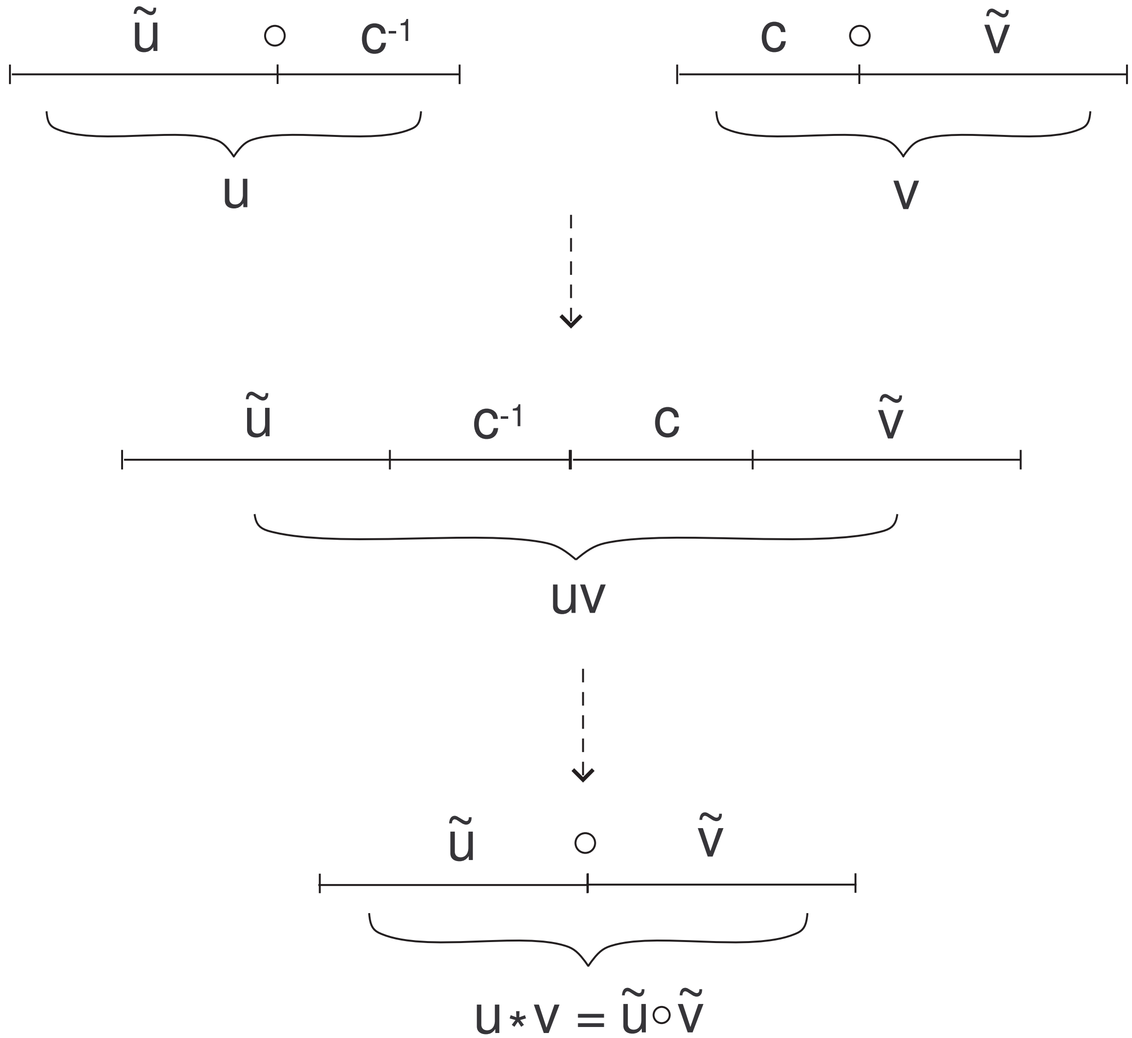
Suppose u and v can be represented in the form

$$u = \tilde{u} \circ c^{-1}, v = c \circ \tilde{v},$$

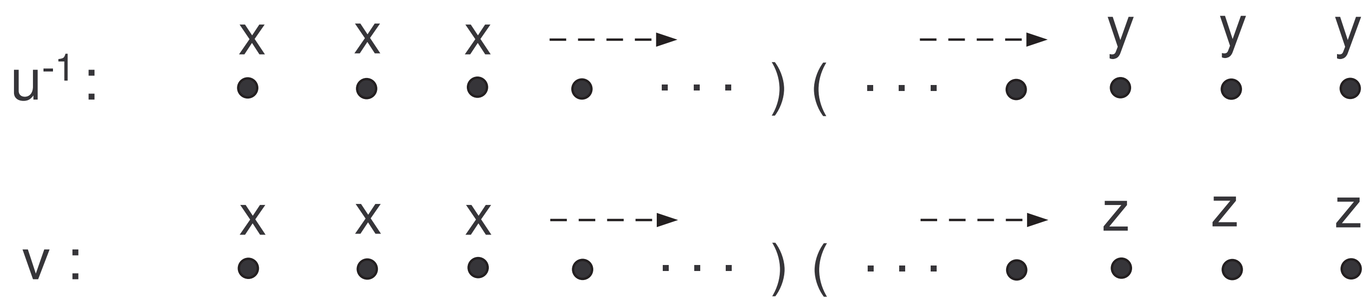
where $c \in R(A, X)$ is of maximal possible length.

Then define

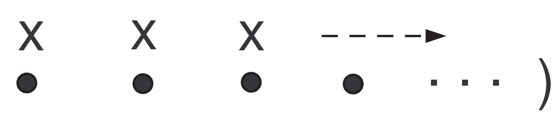
$$u * v = \tilde{u} \circ \tilde{v}.$$


ũ ○ c-1
c ○ ṽ
u
v
ũ c-1 c ṽ
uv
ũ ○ ṽ
u ∗ v = ũ ○ ṽ

Example (product is not defined). $u, v \in R(\mathbb{Z}^2, X)$



The common initial part of u^{-1} and v is



which is not defined on a closed segment. Hence, $u * v$ is not defined.

Cyclic decomposition

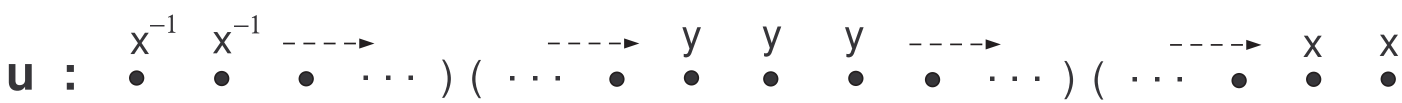
$v \in R(A, X)$ is cyclically reduced if $v(1_A)^{-1} \neq v(|v|)$.

$v \in R(A, X)$ admits a cyclic decomposition if

$$v = c^{-1} \circ u \circ c,$$

where $c, u \in R(A, X)$ and u is cyclically reduced.

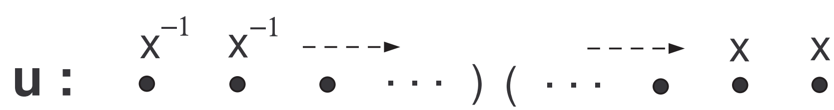
Example. $u \in R(\mathbb{Z}^2, X)$ does not admit a cyclic decomposition



Torsion

$R(A, X)$ has elements of order 2.

Example. $u \in R(\mathbb{Z}^2, X)$



has order 2.

Fact. Let $u \in R(A, X)$. If $u * u$ is defined then either u admits a cyclic decomposition (thus, has infinite order), or has order 2.

$$R^*(A, X) = \{w \in R(A, X) \mid w * w \text{ is defined and not equal to } \varepsilon\}$$

Theorem. (Myasnikov-Remeslennikov-S) If G is a group and there exists an embedding of G into $R^*(A, X)$ then G has a free Lyndon length function with values in A.

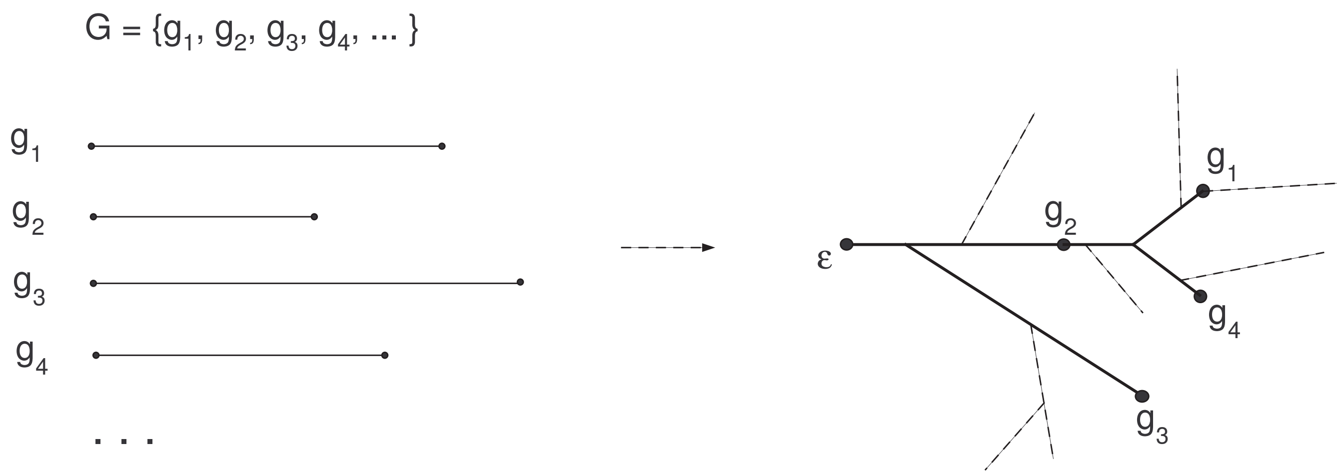
Theorem. (Chiswell) Let G have a free Lyndon length function in an ordered abelian group Λ. Then there exists a discretely ordered abelian group A such that G is embeddable into $R^*(A, X)$ for some set X. Here, $A = \Lambda$ if Λ is discretely ordered, and $A = \mathbb{Z} \oplus \Lambda$ with the right lexicographic order otherwise.

From infinite words back to actions

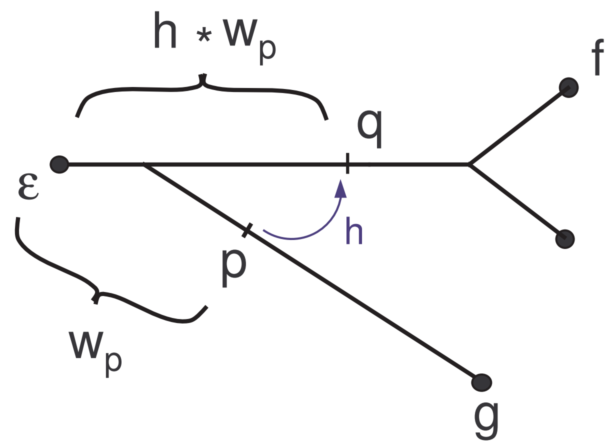
Let $G \hookrightarrow R^*(\Lambda, X)$ for some Λ. Then every $g \in G$ can be viewed as an infinite word.

$\forall\ g_1, g_2 \in G$ the common initial subword $com(g_1, g_2)$ of g_1 and g_2 is defined $\Longrightarrow$ identify g_1 and g_2 by $com(g_1, g_2)$.

As a result we obtain a Λ-tree Γ_G in which every point (except for ε) is labeled by a letter from $X^{\pm}$.



Every point $p \in \Gamma_G$ is connected to ε by a unique geodesic $[\varepsilon, p]$. By definition of Γ_G we can view $[\varepsilon, p]$ as a reduced word w_p, which is an initial subword of some $g \in G$.



$h * w_p$ is defined for any $h \in G$ and $h * w_p$ is an initial subword of some $f \in G$. The terminal endpoint q of $h * w_p$ belongs to Γ_G and we define the action

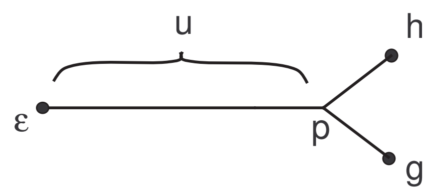
$$h \cdot p = q$$

The action defined above is isometric.

Regular length functions (actions)

Let $G \hookrightarrow R^*(\Lambda, X)$ and consider Γ_G constructed above.

$p \in \Gamma_G$ is a branch point if it is the terminal endpoint of $com(g, h)$, $g, h \in G$.



The underlying length function on G (or the action of G on Γ_G) is regular if

$$u = com(g, h) \in G \text{ for all } g, h \in G.$$

regularity $\Longrightarrow$ there is only one G-orbit of branch points in Γ_G

Example. Let $F(x, y)$ be a free group and $H = \langle x^2y^2,\ xy \rangle$ be its subgroup.

F has natural free $\mathbb{Z}$-valued length function $l_F : f \longrightarrow |f|$. Hence, l_F induces a length function l_H on H.

l_F is regular, but l_H is not

Take $g = xy^{-1}x^{-2},\ h = xy^{-1}x^{-1}y$ in F. Then

$$g, h \in H, \quad \text{but} \quad com(g, h) = xy^{-1}x^{-1} \notin H.$$

Motivation for studying regular free length functions:

- Algebraic Geometry over groups with regular free length functions. Makanin-Razborov process for such groups.

- Nielsen cancellation techniques.

- Stallings' foldings for groups with regular free length functions.

- An approach to description of groups acting on Λ-trees.

Regular completions

Theorem. (Kharlampovich-Myasnikov-Remeslennikov-S) G has a regular free length function in $\mathbb{Z}^n$ if and only if G can be represented as a union of a finite series of groups

$$G_1 < G_2 < \cdots < G_n = G,$$

where

1. G_i has a regular free length function in $\mathbb{Z}^i$ (that is, G_1 is a free group),
2. G_{i+1} is obtained from G_i by finitely many HNN-extensions in which associated subgroups are maximal abelian and length-isomorphic.

Let G be a f.g. Λ-free group with the length function $|\cdot|_G : G \to \Lambda$.

Question: Does there exist a f.g. Λ-free group G^* (with the length function $|\cdot|_{G^*} : G^* \to \Lambda$) and an embedding $\phi : G \to G^*$ such that $|\phi(g)|_{G^*} = |g|_G$ for each $g \in G$?

We call G^* above a regular completion of G.

Existence of a regular completion in the case $\Lambda = \mathbb{Z}$ easily follows from the fact that every $\mathbb{Z}$-free group is free. Indeed, if G is free then its elements can be viewed as finite words over some alphabet X which can be assumed finite since G is f.g. Thus, G embeds into $F(X)$.

In fact, a more interesting regular completion may be constructed, which besides length preserves more properties of the action of G on its associated tree Γ_G.

Proposition. Let G be a subgroup of $R^*(\mathbb{Z}, X)$. Then there exists a finite alphabet Y and an embedding $\phi : G \to H$, where $H = F(Y)$, inducing an embedding $\psi : \Gamma_G \to \Gamma_H$ such that

(i) $|g|_G = |\phi(g)|_H$ for every $g \in G$,

(ii) if A is a maximal abelian subgroup of G then $\phi(A)$ is a maximal abelian subgroup of H,

(iii) if a and b are non-G-equivalent ends of Γ_G then $\psi(a)$ and $\psi(b)$ are non-H-equivalent ends of Γ_H,

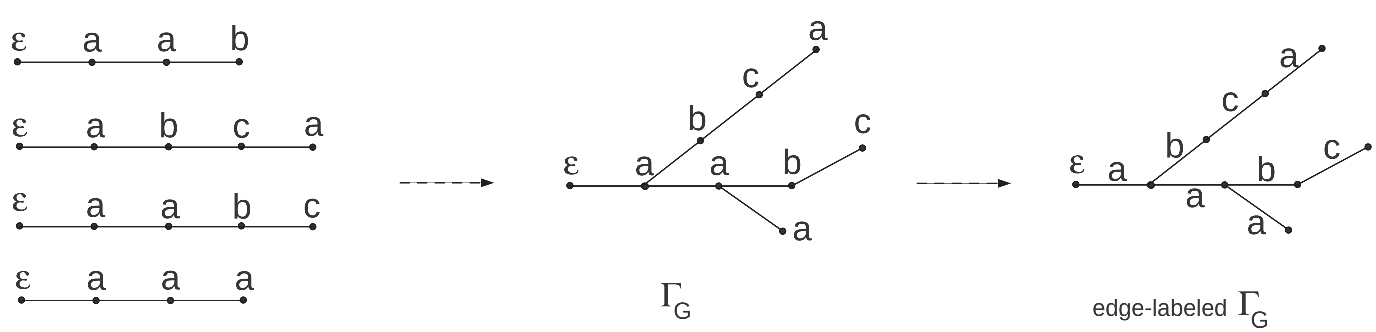
(iv) if A and B are maximal abelian subgroups of G which are not conjugate in G then $\phi(A)$ and $\phi(B)$ are not conjugate in H.

Idea of the proof

Let $G \subset R^*(\mathbb{Z}, X)$ be f.g. Hence, $G \leq F(X)$, where X is finite.

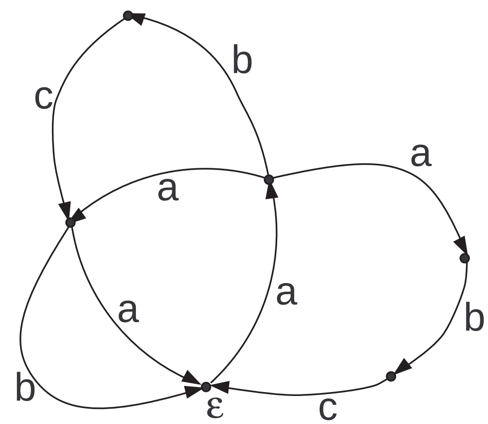
Construct Γ_G.

Example. $G = \langle aab,\ abca,\ aabc,\ a^3 \rangle \leq F(X),\ X = \{a, b, c\}$.



Edge-labeled Γ_G can be viewed as a subtree of the Cayley graph of $F(X)$.

Next, consider Γ_G/G, which is a finite X-labeled folded graph with a base-point ε.

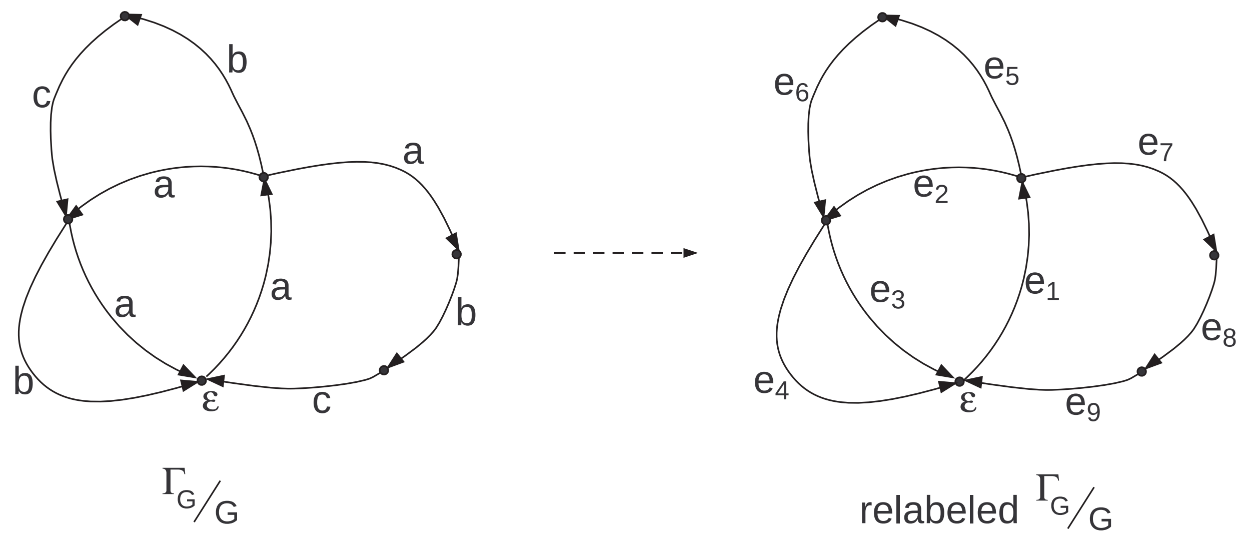


Γ_G/G

Observe that the subgroups $H_1 = \langle a^3 \rangle,\ H_2 = \langle abca \rangle, H_3 = \langle aabc \rangle$ are maximal cyclic in G, moreover H_2 and H_3 are not conjugate in G.

Now, the embedding $\theta : G \hookrightarrow F(X)$ preserves the length but $\theta(H_1)$ is not maximal in $F(X)$, and $\theta(H_2)$ is conjugate to $\theta(H_3)$ in $F(X)$.

Instead, we construct another embedding $\phi : G \hookrightarrow F(Y)$, where Y is the set of positively oriented edges in Γ_G/G. This embedding can be obtained from the relabeling of Γ_G/G as shown below.



That is,

$aab \to e_1e_2e_4,\ abca \to e_1e_5e_6e_3,\ aabc \to e_1e_7e_8e_9,\ a^3 \to e_1e_2e_3.$

Fact. ϕ satisfies all the properties stated in Proposition.

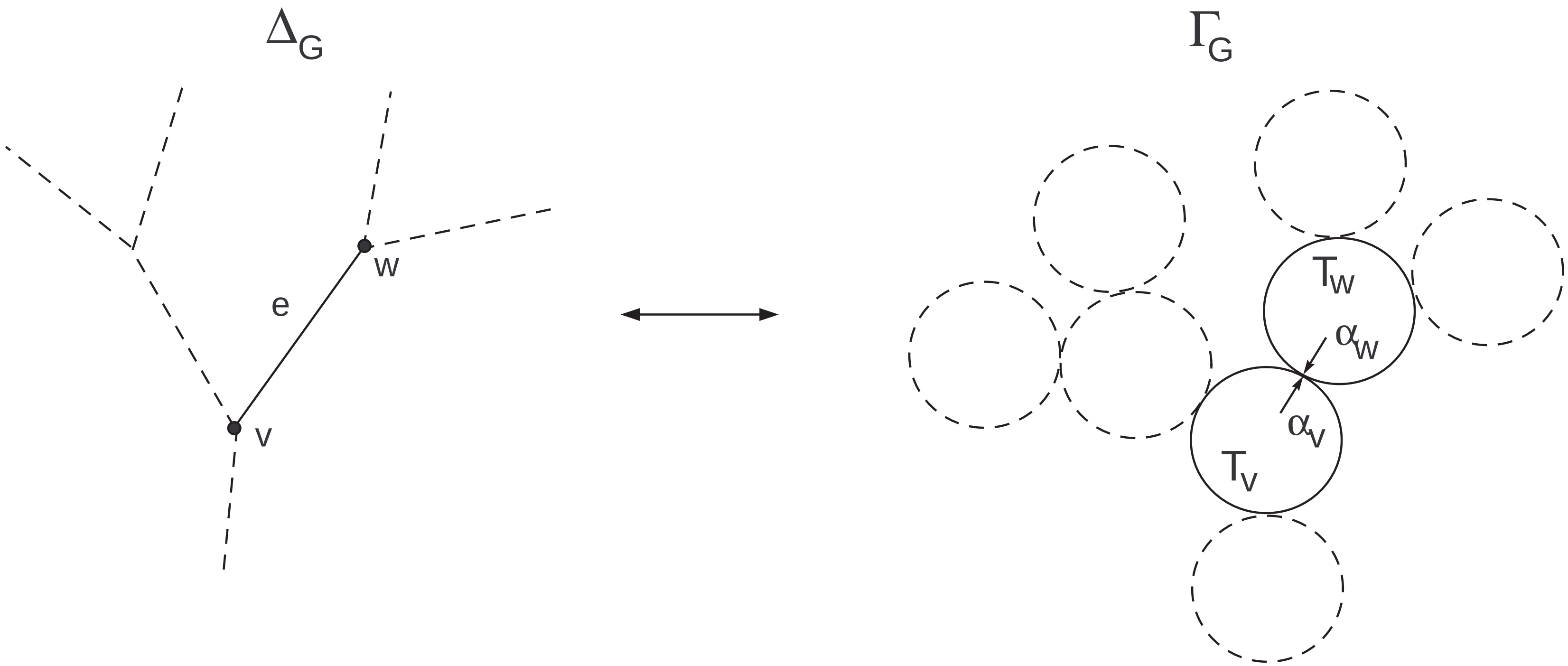
General case: $\Lambda = \mathbb{Z}^n$

Let $G \subset R^*(\mathbb{Z}^n, X)$ be finitely generated. Construct Γ_G.

Consider $\Delta_G = \Gamma_G / \sim$, where "$\sim$" is an equivalence of $\mathbb{Z}^{n-1}$-close points in Γ_G. Hence, Δ_G is a simplicial tree on which G acts.

Every vertex v of Δ_G corresponds to a $\mathbb{Z}^{n-1}$-subtree T_v of Γ_G. The stabilizer $Stab_G(v)$ of v in Δ_G is the stabilizer of T_v in Γ_G, hence, it is a $\mathbb{Z}^{n-1}$-free group.

Every edge $e = (v, w)$ of Δ_G corresponds to a pair of ends (α_v, α_w) of T_v and T_w. The stabilizer $Stab_G(e)$ of e is a subgroup of $\mathbb{Z}^n$ which embeds into $Stab_G(v)$ and $Stab_G(w)$. Its copies are maximal abelian subgroups which stabilize α_v and α_w respectively in T_v and T_w.


Δ_G
w
e
v
Γ_G
T_w
α_w
α_v
T_v

By Bass-Serre Theory, G can be constructed from finitely many vertex stabilizers of Δ_G using finitely many free products with amalgamation and HNN-extensions along the edge stabilizers of Δ_G.

Let H_1, H_2 be $\mathbb{Z}^{n-1}$-free groups (vertex stabilizers).

Let $A \leq H_1$ and $B \leq H_2$ be length-isomorphic maximal abelian subgroups. Hence, $H = H_1 *_{A=B} H_2$ is also $\mathbb{Z}^{n-1}$-free.

Suppose there exist length-preserving embeddings

$$\phi_1 : H_1 \hookrightarrow K_1, \ \phi_2 : H_2 \hookrightarrow K_2,$$

where K_1, K_2 have free regular length functions in $\mathbb{Z}^{n-1}$, and $\phi_1(A), \phi_2(B)$ are still maximal abelian respectively in H_1 and H_2.

How to embed $H = H_1 *_{A=B} H_2$ into a group with a free regular length function ?

Algebraically it can be done as follows.

Step 1.

$$H = H_1 *_{A=B} H_2 \hookrightarrow H^* = \langle H_1 * H_2,\ s \mid s^{-1}As = B\rangle$$
$$H \simeq \langle s^{-1}H_1 s,\ H_2\rangle < H^*$$

Step 2.

$$H_1 * H_2 \hookrightarrow K = K_1 * K_2$$

K has a regular free length function (since both K_1 and K_2 have them).

Step 3.

$$H^* \hookrightarrow K^* = \langle K, s \mid s^{-1}\phi_1(A)s = \phi_2(B)\rangle$$

K^* has a regular free length function (since K has it) because $\phi_1(A)$ and $\phi_2(B)$ are maximal abelian in K.

The embedding does not preserve the length on H.

More subtle approach

Let T_1, T_2 be $\mathbb{Z}^{n-1}$-trees for H_1 and H_2.

Assume that for each $i = 1, 2$ there exists an embedding $\phi_i : H_i \hookrightarrow K_i$ such that

(a) K_i has a free regular action on a $\mathbb{Z}^{n-1}$-tree S_i,

(b) $\theta_i : T_i \hookrightarrow S_i$ is an isometric embedding,

(c) θ_i is equivariant under the action of H_i,

(d) if C ia a maximal abelian subgroup of H_i then $\phi_i(C)$ is a maximal abelian subgroup of K_i,

(e) if C and D are maximal abelian subgroups of H_i which are not conjugate in H_i then $\phi_i(C)$ and $\phi_i(D)$ are not conjugate in K_i.

Moreover, we can always assume that $H_i \subset R^*(\mathbb{Z}^{n-1}, Y_i),\ i = 1, 2,$ where $Y_1 \cap Y_2 = \emptyset$.

The conditions (a), (b), (c) just ensure the length preservation under ϕ_i.

The condition (d) guarantees that the stabilizer of an end in T_i remains the stabilizer of an end in S_i.

Finally, the condition (e) makes sure that it is possible to construct $s \in R^*(\mathbb{Z}^n, Y_1 \cup Y_2)$ so that $\langle K_1, K_2, s\rangle$ forms a group in $R^*(\mathbb{Z}^n, Y_1 \cup Y_2)$ and $H = H_1 *_{A=B} H_2$ embeds (preserving length) into $\langle K_1, K_2, s\rangle$. Moreover, the length function on $\langle K_1, K_2, s\rangle$ is regular.

Theorem. (Kharlampovich-Myasnikov-S) Every f.g. group with a free $\mathbb{Z}^n$-valued length function can be embedded into a f.g. group with a regular free $\mathbb{Z}^n$-valued length function. Moreover, this embedding preserves length.