Efficient Enumeration of Cosets.

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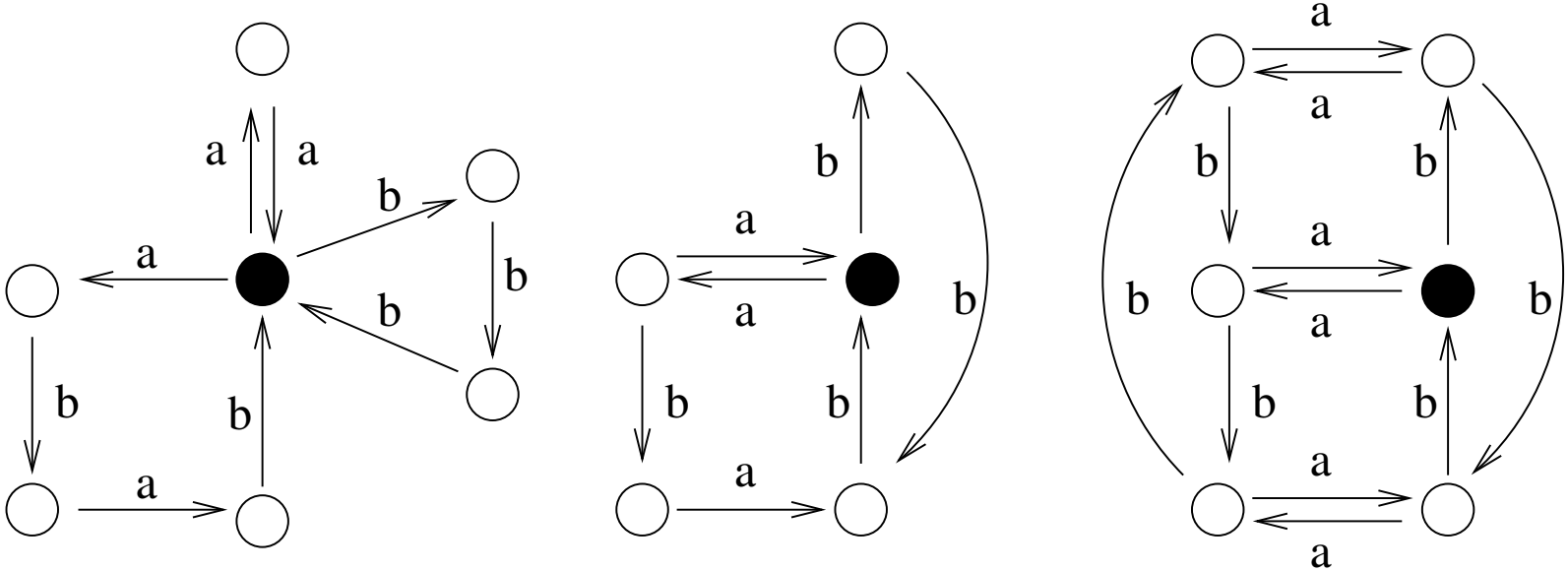

Time complexity

Definition For any presentation P, R is the sum of the lengths of all cyclic conjugates of relators.

The first pass adds at most R new cosets, i.e., vertices.

The second pass adds at most R^2 etc.

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After p passes we have at defined at most $O(R^{p+1})$ cosets.

The time complexity is $O(R^{p+1})$ or perhaps $O(R^{3(p+1)})$ taking into account the resolution of coincidences.

We will measure time by the number of passes.

Estimating the number of passes

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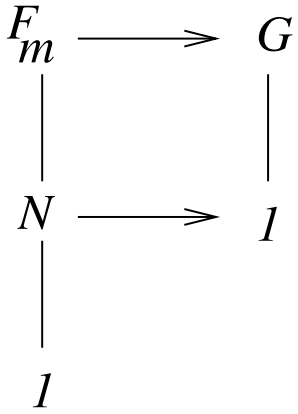

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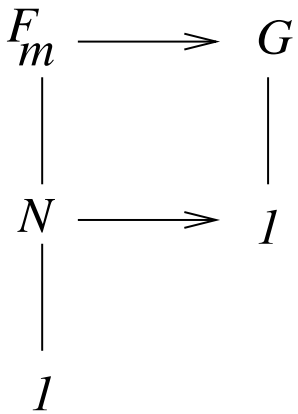

N is generated by finitely many conjugates of the r_j's.

Lemma If N is generated by $\{r_j^w \mid |w| < p\}$, then coset enumeration succeeds in at most $p + 1$ passes.

Worst case complexity

$\{a_1, a_2, \ldots\}$ is a denumerable set of generators.

A presentation has the form $P = \langle a_1, \ldots, a_m \mid r_i \rangle$.

I is the set of all presentations.

J is the subset of I which defines finite groups.

$T(P)$ is the number of passes for $P \in J$.

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$f(n)$ is not computable.

If it were, we could decide whether or not $P \in I$ defined a finite group.

Generic case complexity

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I. Kapovich, A. Myasnikov, P. Schupp, V. Shpilrain, Generic-case complexity and decision problems in group theory, J. of Algebra, 264 (2003), 665-694.

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The Simplex Algorithm has instances which require exponential time, but these instances are never encountered in practice.

Smale and Vershik and Sporyshev showed that the set of exceptional instances has measure 0.

Generic properties of groups

A generic set is the complement of a negligible set.

If G has an infinite cyclic quotient, then the word problem is solvable in linear time on a generic set of inputs. (KMSS)

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Our approach to coset enumeration follows Ushakov.

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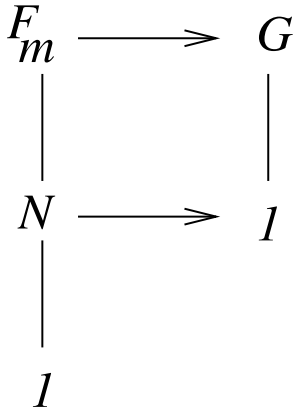
Generic complexity of coset enumeration

J is the set of finite presentations of finite groups.

We work with a covering $\tilde{J} \longrightarrow J$.



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$P = \langle a_1, \ldots, a_m \mid r_j \rangle$

$r_j \in N$

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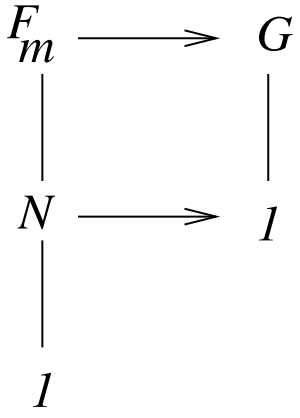

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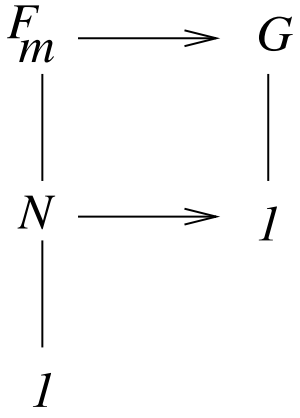

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with all relators cyclicly reduced.

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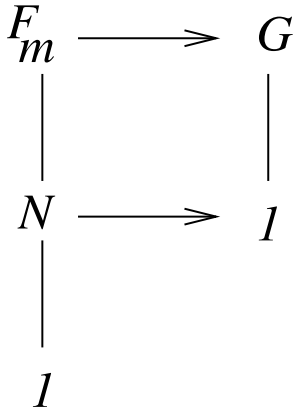

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with all relators cyclicly reduced.

Lemma Every $P \in J$ is obtained in this way.

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Lemma $\tilde{J}$ maps onto J, and P presents F_m/N.

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Proof Coset enumeration. Every $r_p^{u_p}$ can eventually be traced at 1.

Hence each $W_k(r_p^{u_p}, \ldots)$ can be traced.

Since $x_k W_k(r_p^{u_p, \cdots})^{-1}$ is a relator, x_k is traceable.

All relators of P lie in N and every freely reduced word in N is traceable. $\square$

The number of passes is the maximum of the lengths of the u_p's plus 1.

The size function

$$(m, N, \{x_k\}, \{W_k(x_p, y_p)\}, \{r_j\}, \{u_t\}) \to \langle a_i \mid r_j, x_k(W_k(r_p^{\pm u_p})^{-1} \rangle$$

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The length of the r_j's is as words in the Schreier generators of N.

When they are rewritten as words in F_m, they do not get any shorter.

J_n is the image of $\tilde{J}_n$

The probability distributions

The probability distribution μ_n is induced by $\tilde{\mu}_n$ on $\tilde{J}_n$.

$\tilde{J}_n = \cup_{m,N}(m, N, \{x_k\}) \times \{(\{W_k(x_p, y_p)\}, \{r_j\}, \{u_t\})\}$

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Theorem Let L be the maximum relator length. For presentations in J, coset enumeration succeeds in at most $1.2L$ passes on a generic set of presentations.

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3. Where are the difficult presentations? Perhaps we should look at presentations with short relators.
4. On the other hand B. H. Neumann found a series of presentations P_n of the trivial group with 3 generators, 3 relators and $L = 5^n$. The generation of this sequence seems somewhat similar to the product replacement procedure for generating random group elements. Are these P_n's random presentations? Does coset enumeration converge in $1 + 5^n$ passes on P_n?