

# Nonlinearity of matrix groups

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Recall that for every (associative unitary) ring  $R$  the group  $\text{EL}_k(R)$  is the subgroup of  $\text{GL}_k(R)$  generated by all  $k \times k$ -elementary matrices  $x_{ij}(r) = \text{Id} + re_{ij}$  for  $r \in R$ ,  $1 \leq i \neq j \leq k$ .

The group  $\text{EL}_k(\mathbb{Z}\langle x_1, \dots, x_s \rangle)$  surjects onto any  $\text{EL}_k(R)$  for  $s$ -generated rings  $R$ . This implies that  $\text{EL}_k(\mathbb{Z}\langle x_1, \dots, x_s \rangle)$  are residually finite. Note that these groups are finitely generated (Milnor) and finitely presented (Krstić-McCool) for  $k \geq 4$ . They satisfy property (T) (Ershov-Jajkin-Zapirain, Kassabov).

Clearly, if  $R$  has a faithful  $n$ -dimensional linear representation over a field, then the group  $\text{EL}_k(R)$  has a faithful  $nk$ -dimensional linear representation over the same field.

There are many results about isomorphisms between various matrix groups over (mostly

commutative) rings from the original results of Mal'cev '61 to results of O'Meara '74 to Mostow '73 rigidity results.

There are also many results about embeddings of one matrix group into another. Churkin, '68, proved that the wreath product  $\mathbb{Z} \wr \mathbb{Z}^n$  embeds into a matrix group over a field  $K$  of characteristic 0 if and only if the transcendence degree of  $K$  over its prime subfield is at least  $n$ . Hence  $SL_n(K)$  cannot embed into  $SL_m(K')$  if  $K, K'$  are fields of characteristic 0 and the transcendence degree of  $K$  is bigger than the transcendence degree of  $K'$ . Much stronger non-embeddability results for general linear groups over fields follow from Borel-Tits, '73.

We can also mention the result of Chen '84 who proved that if  $D$  is a division algebra and  $SL_3(D)$  maps into a non-trivial simple algebraic

group  $G(K)$  with dense image, then  $D$  is finite dimensional over its center.

One of the latest results of this sort is the result of Talia Fernos who proved that  $SL_3(\mathbb{Z}[x])$  is not linear over  $\mathbb{Q}$ .

**Theorem 1.** *(a) Let  $R$  be an associative unitary ring,  $k \geq 3$ . The group  $EL_k(R)$  has a faithful finite dimensional representation over  $\mathbb{C}$  if and only if  $R$  has a finite index ideal  $I$  that admits a faithful finite dimensional representation over  $\mathbb{C}$ .*

*(b) The group  $EL_k(\mathbb{Z}\langle x, y \rangle)$ ,  $k \geq 3$ , does not have a faithful finite dimensional representation over any field.*

Part (a) of Theorem 1 does not hold if we replace  $\mathbb{C}$  by a field of positive characteristic.

Note that the “finite index” condition in part (a) of Theorem 1 is necessary, because there are finite rings (say, the endomorphism ring  $R$  of the Abelian group  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}$ , where  $p$  is a prime) that do not have any finite dimensional faithful representations over fields and even over any commutative rings (Bergman, '74). However, the group  $EL_k(R)$  is finite and therefore linear.

Also, parts (a) and (b) of Theorem 1 do not hold for  $EL_2$ . Indeed, by Exercise 2 of Section 2.7 in Cohn “Free rings and their relations”, the subgroup of  $GE_2(\mathbb{Q}\langle x, y \rangle)$  generated by the elementary and diagonal matrices of  $GL_2(\mathbb{Q}\langle x, y \rangle)$  is isomorphic to the group  $GE_2(\mathbb{Q}[x])$  which is a subgroup of  $GL_2(\mathbb{C})$ , and so the group  $EL_2(\mathbb{Z}\langle x, y \rangle) < GE_2(\mathbb{Q}\langle x, y \rangle)$  is linear.

Let  $\pi : \text{EL}_k(R) \rightarrow \text{GL}_n(K)$  be a linear representation of the group  $\text{EL}_k(R)$ , where  $K$  is an algebraically closed field,  $k \geq 3$ .

**Definition 2.** We will denote the set  $\{\pi(x_{13}(r)) \mid r \in R\}$  by  $U$  and its Zariski closure by  $V$ . By construction  $V$  is an algebraic variety.

**Theorem 3.** There exist two distinguished elements  $\mathbf{0}$  and  $\mathbf{1}$  in  $V$  and polynomial maps  $+, \times : V \times V \rightarrow V$ ,  $- : V \rightarrow V$ , which give  $V$  a structure of an associative ring. Moreover, the map  $\rho : R \rightarrow U \subset V$  defined by  $\rho(r) = \pi(x_{13}(r))$  is a ring homomorphism.

*Proof.* The addition  $+$  :  $U \times U \rightarrow U$  as follows  $u_1 + u_2 := u_1 u_2$  (the extension of that polynomial map to  $V$ ).

In order to define the “multiplication” we need to use two special elements  $w_{23}$  and  $w_{12}$  in

$EL_k(R)$  which have the properties

$$w_{12}x_{13}(r)w_{12}^{-1} = x_{23}(r) \text{ and}$$

$$w_{23}x_{13}(r)w_{23}^{-1} = x_{12}(r).$$

$$w_{12} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad w_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Now we can define the algebraic map  $\times : U \times U \rightarrow GL_n(K)$  as follows

$$u_1 \times u_2 := [\pi(w_{23})u_1\pi(w_{23})^{-1}, \pi(w_{12})u_2\pi(w_{12})^{-1}].$$

The commutator relation

$$[x_{12}(r), x_{23}(s)] = x_{13}(rs)$$

implies that

$$\rho(r_1) \times \rho(r_2) = \rho(r_1 \cdot r_2),$$

thus  $\times$  is a map from  $U \times U$  to  $U$  and can be extended to a polynomial map from  $V \times V$  to  $V$ . The element  $\rho(1)$  plays the role of the unit with respect to this multiplication and we will call it  $\mathbf{1} \in V$ .

Thus  $\mathbf{0}, \mathbf{1}$  and the maps  $+$  and  $\times$  turn  $V$  into an associative ring with a unit.  $\square$

**Lemma 4.** *Let  $V$  be an algebraic variety with two algebraic operations given by polynomial functions, that turn it into an associative ring with  $\mathbf{1}$ . Let  $V_0$  denote the connected component of  $\mathbf{0}$  in  $V$ . Then:*

(a)  $V_0$  is a two-sided ideal in  $V$ ;

(b) the quotient  $V/V_0$  is a finite ring.

*Proof.* a) The set  $xV_0$  is connected and contains  $\mathbf{0}$ . Hence  $xV_0 \subseteq V_0$  since  $\mathbf{0}$  is non-singular.

b) The number of components is finite.  $\square$

**Lemma 5.** *Let  $V$  be an algebraic variety over  $\mathbb{C}$  with two algebraic operations, that turn it into an associative ring with 1. Then the irreducible component  $V_0$  of  $\mathbf{0}$  in  $V$  is isomorphic to a finite dimensional algebra over  $\mathbb{C}$ , i.e., the ring  $V$  is virtually linear over  $\mathbb{C}$ .*

*Proof.* Note that the additive group  $V_+$  of  $V$  is an Abelian Lie group over  $\mathbb{C}$ .

It is well known (Pontryagin), that  $V_0$  is a product of a finite number of copies of  $\mathbb{C}$  and a finite number of 1-dimensional tori. Therefore, the fundamental group  $\Gamma$  of  $V_0$  (based at  $\mathbf{0}$ ) is isomorphic to  $\mathbb{Z}^k$  for some  $k < \infty$ , and the product of any two loops in  $\Gamma$  is homotopic to their point-wise sum in  $V_0$ .

Multiplication by an element in  $V$  induces an endomorphism of  $\mathbb{Z}^k$  and so we have a map  $\phi$  from  $V$  to the endomorphism ring  $\text{End}(\mathbb{Z}^k)$ . This map is continuous and a ring homomorphism.

The endomorphism ring  $\text{End}(\mathbb{Z}^k) = GL_k(\mathbb{Z})$  is discrete, therefore the image of  $V_0$  is trivial and  $\phi$  factors through a map  $\bar{\phi} : V/V_0 \rightarrow \text{End}(\mathbb{Z}^k)$ . Since  $\phi(1) = 1$ , either  $k = 0$  or  $\text{End}(\mathbb{Z}^k)$  contains a finite non-trivial subring. The latter option is impossible.

Thus the fundamental group  $\Gamma$  is trivial and  $V_0$  is a simply connected Abelian Lie group over  $\mathbb{C}$ . Therefore,  $V_0$  is isomorphic to a finite dimensional vector space over  $\mathbb{C}$ . The distributive laws imply that multiplication on  $V_0$  is bilinear, i.e.,  $V_0$  is a finite dimensional algebra over  $\mathbb{C}$ . □

**Corollary 6.** *Let  $V$  be an algebraic variety over a field of characteristic 0 with two algebraic operations, that turn it into an associative ring with 1. Then any ring homomorphism  $\phi : \mathbb{Z}\langle x, y \rangle \rightarrow V$  has a non-trivial kernel.*

**Corollary 7.** (Chen) *Let  $D$  be a non-commutative division ring of characteristic 0. Then the group  $\text{EL}_n(D)$ ,  $n \geq 3$ , has nontrivial finite dimensional representations in characteristic 0, if and only if  $D$  is finite dimensional over its center.*

*Proof.* Let  $\pi : \text{EL}_k(D) \rightarrow \text{GL}_n(K)$  be a nontrivial representation of the group  $\text{EL}_k(D)$ . The group  $\text{EL}_k(D)$  is normally generated by the subgroup  $\{x_{13}(d) \mid d \in D\}$ , therefore the map  $\rho : D \rightarrow U$  is a ring homomorphism with a non-trivial image. Thus,  $\rho$  has to be an isomorphism, since  $D$  does not have any non-trivial ideals.

By Lemma 5, the division ring  $D$  is linear, therefore it satisfies some polynomial identity. Finally, by a theorem of Kaplansky every division algebra  $D$  satisfying a polynomial identity is finite dimensional over its center. The other direction is obvious.  $\square$

**Remark 8.** *Let  $K$  be an infinite field of characteristic 2 and let  $V = K \times K$  with the following operations:*

$$(a, b) + (c, d) = (a + c, ac + b + d),$$

$$(a, b) \times (c, d) = (ac, bc^2 + a^2d).$$

*One can verify directly that  $V$  is a commutative ring.*

*The elements  $(0, b)$  form an ideal  $I$  with zero multiplication,  $V$  is a local ring with the maximal ideal  $I$ . Therefore, that ring does not have proper ideals of finite index.*

*Since all elements of the form  $(a, b)$ ,  $a \neq 0$ , have additive order 4, this ring is not linear over any field but it is an algebraic variety over  $K$ .*

**Lemma 9.** *Let  $V$  be an algebraic variety over a field  $K$  (of arbitrary characteristic) with two algebraic operations, that turn it into an associative ring with 1. If  $V$  is irreducible then the multiplicative group of  $V$  is linear over  $K$ .*

**Remark 10.** Let  $V$  be the variety with the ring structure constructed in Remark 8. The group  $\text{EL}_3(V)$  is a subgroup of the multiplicative semigroup of the ring of  $3 \times 3$  matrices over  $V$ , which is an algebraic variety (isomorphic to  $K^{18}$ , where the addition and the multiplication are given by some polynomial functions of degree at most 4). Lemma 9 implies that  $\text{EL}_3(V)$  is a linear group over  $K$ . Thus, there exists a

ring  $R$  which is not (virtually) linear over any field, but the group  $EL_3(R)$  is linear. Hence, part (a) of Theorem 1 does not hold in the case of characteristic 2. This construction can be generalized to any positive characteristic.

**Theorem 11.** *Let  $V$  be an algebraic variety with two algebraic operations which turn it into an associative ring. Then any ring homomorphism  $\phi : \mathbb{Z}\langle x, y \rangle \rightarrow V$  has a non-trivial kernel.*

*Proof.* Let  $k$  be the dimension of  $V$  and let assume that the map  $\phi$  is injective. Let  $s_l$  denote the anti-symmetric polynomial function on  $l$  arguments, i.e.,

$$s_l(x_1, \dots, x_l) = \sum_{\sigma \in S_l} (-1)^\sigma \prod x_{\sigma(i)}.$$

Pick elements  $r_1, r_2, \dots, r_{k+1}$  such that  $s_l(r_1, \dots, r_l)$  is not 0 in the ring  $R = \mathbb{Z}\langle x, y \rangle$  for any  $l \leq k+1$

(for example we can take  $r_i = xy^{i+1}$ ). Let  $M_l$  denote the  $\mathbb{Z}$ -span of the elements  $r_1, \dots, r_l$  in the ring  $R$  and let  $N_l$  be the Zariski closure of  $\phi(M_l)$  in  $V$ .

**Lemma 12.** *For any  $l$  we have that  $\dim N_l > \dim N_{l-1}$ .*

*Proof.* Let  $N_{l,i}$  denote the set  $i \cdot \phi(r_l) + N_{l-1}$  for a positive integer  $i$  (here  $i \cdot r$  denotes the sum  $r + r + \dots + r$ ).

These are algebraic subvarieties of  $N_{l-1}$  which are of the same dimension as  $N_{l-1}$  and disjoint.

□

This is a contradiction, because by construction  $N_{k+1} \subset V$ .

**Remark 13.** It is not clear if it is possible to embed  $\mathbb{F}_p\langle x, y \rangle$  into an algebraic variety with a

ring structure over a field of positive characteristic. (The above argument only works if the “base ring” contains  $\mathbb{Z}$ .)

**Definition 14.** *The Steinberg group  $St_k(R)$  of an associative unitary ring  $R$  is the group has (formal) generators  $x_{ij}(r)$  for  $1 \leq i \neq j \leq k$  and  $r \in R$ , which satisfy the following commutator relations:*

$$\begin{aligned} x_{ij}(r)x_{ij}(s) &= x_{ij}(r + s) \\ [x_{ij}(r), x_{pq}(s)] &= 1 && \text{if } i \neq q, j \neq p \\ [x_{ij}(r), x_{pq}(s)] &= x_{ik}(rs) && \text{if } i \neq q \text{ and } j = p. \end{aligned}$$

There is a surjection from  $St_k(R)$  onto  $EL_k(R)$  mapping  $x_{ij}(r)$  to  $\text{Id} + re_{ij}$ . The kernel of this surjection is denoted by  $K_{2,k}(R)$ .

**Lemma 15.** *If  $R$  is a finite ring, then the Steinberg group  $St_k(R)$  is finite for any  $k \geq 3$ .*

*Proof.* Using results of Vaserstein and Milnor we deduce that in the case of a finite ring  $R$ , the groups  $K_{2,k}(R)$  are finite, do not depend on  $k \geq 3$  and are central in  $\text{St}_k(R)$  (because the stable range of a finite ring is 1). Therefore, the Steinberg group  $\text{St}_k(R)$ ,  $k \geq 3$ , is an extension of a finite group by a finite group and thus is also finite.  $\square$

*Proof of Theorem 1.* (a) Suppose that  $G = \text{EL}_k(R)$  is linear over  $\mathbb{C}$ . Then by Theorem 3  $R$  embeds into a ring that is a variety over  $\mathbb{C}$ . By Lemmas 4 and 5, the ring  $R$  has a finite index ideal that is linear over  $\mathbb{C}$ .

Suppose now that  $R$  has a finite index ideal  $I$  that is linear over  $\mathbb{C}$ . Consider the *congruence subgroup*  $G_I$  of  $G$  corresponding to  $I$ , that is the subgroup generated by all  $x_{ij}(r)$ ,  $r \in I$ . The subgroup  $G_I$  has a finite index in  $G$ , because the quotient  $G/G_I$  is a homomorphic image of the Steinberg group  $\text{St}_n(R/I)$  which is

finite by Lemma 15. Also,  $G_I$  is linear over  $\mathbb{C}$ , therefore  $G$  is linear over  $\mathbb{C}$  (consider the representation induced by the faithful representation of  $G_I$ ).

(b) Suppose that  $G = \text{EL}_k(\mathbb{Z}\langle x, y \rangle)$  is linear over a field  $K$ . Again, by Theorem 3, the ring  $\mathbb{Z}\langle x, y \rangle$  embeds into a ring that is a finite dimensional algebraic variety over  $K$ . By Theorem 11, that is impossible, a contradiction.  $\square$

**Remark 16.** *The argument in the proof of part a), work over any field, i.e., if the ring  $R$  has an ideal  $I$  of finite index with a faithful linear representation then the group  $\text{EL}_k(R)$  is linear for  $k \geq 3$ .*