On the lattice of subgroups of free groups[1]

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[1] joint work with A. Miasnikov (McGill), E. Ventura (Barcelona), P. Silva (Porto)

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- or as a free factor, $H \leq_{\mathrm{ff}} F$, every basis of H can be extended to a basis of F – looks closer to vector spaces

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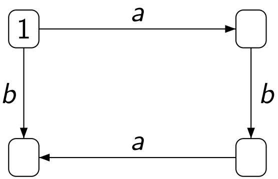

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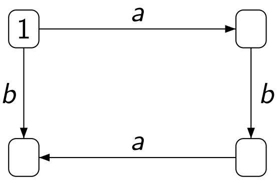

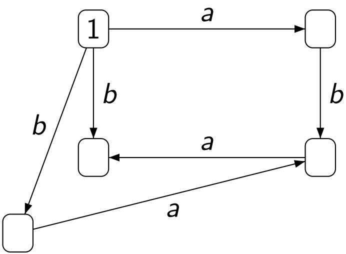

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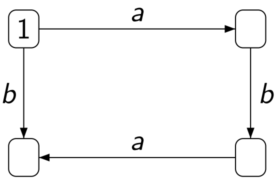

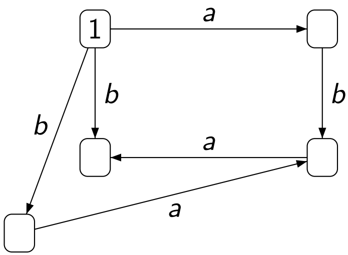

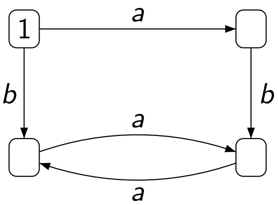

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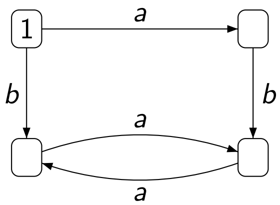

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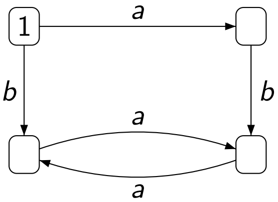
- $H =$ all reduced words that read from 1 to 1



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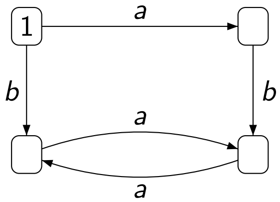

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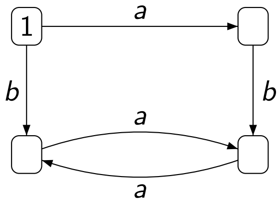
- $\Gamma(H)$ is computable in $O(n \log^* n)$ (Touikan 2006)



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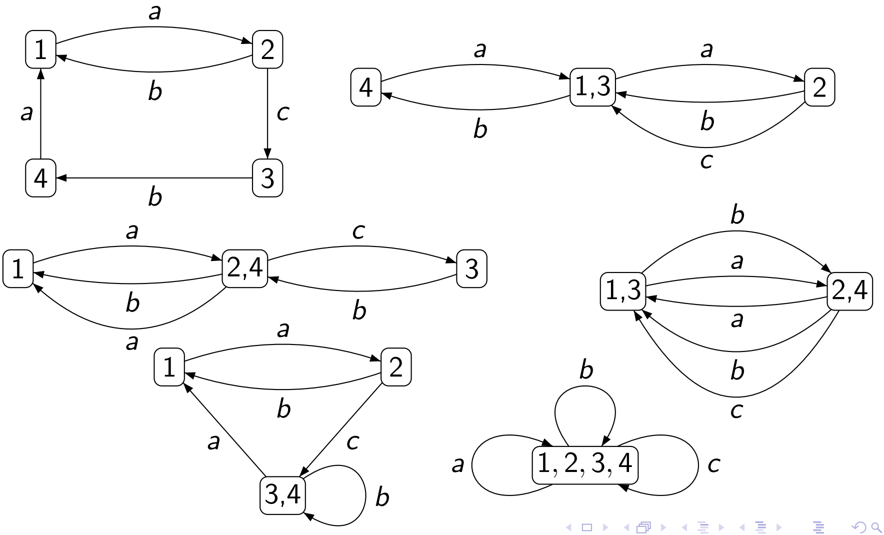
- Computable. May have exponential size.

Example: the six quotients of $\Gamma_A(\langle ab, acba \rangle)$



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Given $H \leq_{\text{fg}} F$, there exists a finite and computable collection $K_1, \ldots, K_n$ of extensions of H, such that whenever $H \leq K \leq F$, then $H \leq K_i \leq_{\text{ff}} K$ for some i.

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- See Ventura 1997, Margolis, Sapir, Weil 2001, Kapovich, Miasnikov 2002

Algebraic extensions

- Say that K is an *algebraic* extension of H, $H \leq_{\text{alg}} K$, if H does not sit in a proper free factor of K: $H \leq L \leq_{\text{ff}} K$ implies $L = K$ (Kapovich, Miasnikov)

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- Question: Is $\text{AE}(H) = \bigcap_A \mathcal{O}_A(H)$?

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- $H \leq_{\text{ff}} F$ if and only if $\text{AE}(H) = \{H\}$

algebraic extensions vs. free multiples: dual notions

Let $H \leq K \leq L$

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- If $H \leq_{\text{fg}} K$, there is also an algebraic closure of H in K, $\text{cl}_K(H)$
- A characterization: let $\varphi \in \text{Aut}(F)$ such that $\varphi(H)$ has minimum size, and let $B \subseteq A$ be the set of letters in $\varphi(H)$: then $\text{cl}(H) = \varphi^{-1}\langle B \rangle$ (Ventura 2009)

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- H is *pure* if $n \neq 0$ and $x^n \in H$ implies $x \in H$: same result as for malnormality

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- The question is open for the pro-solvable closure...

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- Say that $H \leq_{\mathrm{ealg}} K$ if $H \leq_{\mathrm{alg}} K_1 \leq_{\mathrm{alg}} \cdots \leq_{\mathrm{alg}} K_n \leq_{\mathrm{alg}} K$ and each extension is elementary; if $H \leq_{\mathrm{ealg}} K$, then $\mathrm{rank}(K) \leq \mathrm{rank}(H)$

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- The same inequality holds for the pro-$\mathcal{V}$ closure of H: is it an e-algebraic extension?

Finite index and covers

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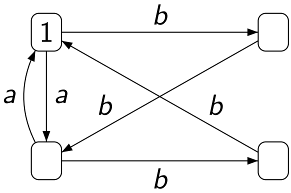

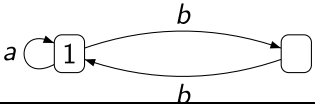

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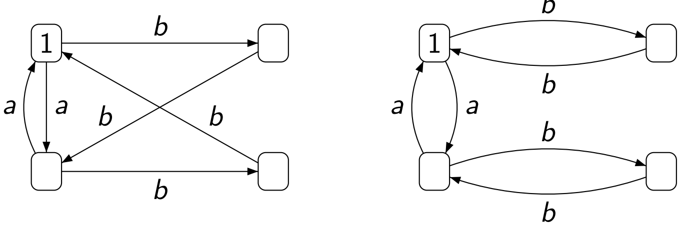

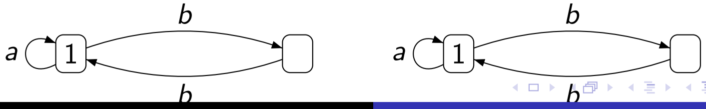

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- H_{fi} is the commensurator of H (all x such that $H \cap H^x \leq_{\text{fi}} H, H^x$)

Computing the finite-index extensions of H (cycl. reduced)

- If p is a vertex of $\Gamma(H)$, let $\tilde{L}_p(H) =$ the language accepted by $\Gamma(H)$ with initial state p, all states final
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Theorem

$\Gamma(H)/(p = q)$ is a finite index extension if and only if $L_p(H) = L_q(H)$ (or $\tilde{L}_p(H) = \tilde{L}_q(H)$)

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if $p \xrightarrow{u} r$, then $L_r(H) = \{\mathrm{red}(\bar{u}x) \mid x \in L_p(H)\}$
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- So if there are at most $f(n)$ finite index extensions, then $f(n) \leq nf(n/2)$, $f(n) \leq n^{\frac{1}{2}(1+\log_2 n)} = \sqrt{n\ 2^{\log_2^2 n}}$

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- To find all these pairs (p, q): minimize the automaton $\Gamma(H)$ with all states final! This can be done in $\mathcal{O}(n \log n)$

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Thank you for your attention!