Group acting on trees

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In this talk I will describe some important techniques that we used to work with fully residually free (limit) groups. We used these techniques (infinite words and so-called elimination processes) in our work on the elementary theory of a free groups. Fully residually free groups appear naturally in geometry, algebra, and logic, though under different names. In particular, they act freely on Z^n -trees. It seems that our methods can provide an adequate tool to attack some open problems concerning with the algebraic structure of finitely generated groups acting freely on Λ-trees. The work is based on joint results with A. Myasnikov and D. Serbin.

The starting point

Theorem (J.-P. Serre, 1980). A group G is free if and only if it acts freely on a tree.

Free action = no inversion of edges and stabilizers of vertices are trivial.

Ordered abelian groups

$\Lambda =$ an ordered abelian group.

Examples:

Archimedean case:

$\Lambda = \mathbb{R}$, $\Lambda = \mathbb{Z}$ with the usual order.

Non-Archimedean case:

$\Lambda = \mathbb{Z}^2$ with the right lexicographic order:

$$(a, b) < (c, d) \iff b < d \text{ or } b = d \text{ and } a < c.$$

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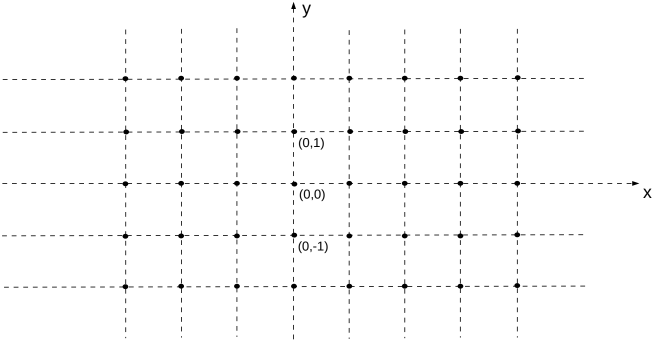
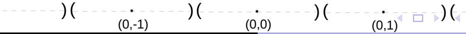
$\mathbb{Z}^2$ with the right-lex ordering



One-dimensional picture



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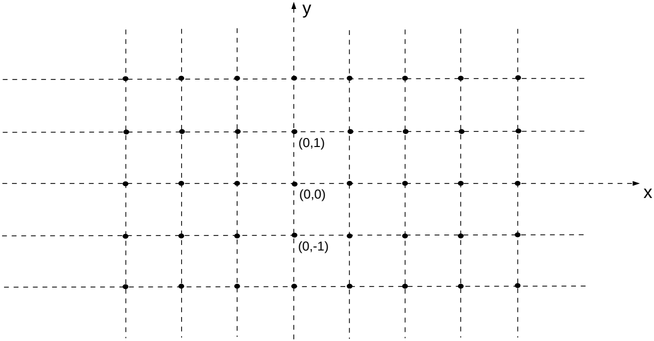

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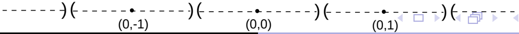

Λ-trees

Morgan and Shalen (1985) defined Λ-trees:
A Λ-tree is a metric space (X, p) (where $p : X \times X \to \Lambda$) which satisfies the following properties:

1) (X, p) is geodesic,
2) if two segments of (X, p) intersect in a single point, which is an endpoint of both, then their union is a segment,
3) the intersection of two segments with a common endpoint is also a segment.

Alperin and Bass (1987) developed the theory of Λ-trees and stated the fundamental research goals:

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Generalize Bass-Serre theory (for actions on $\mathbb{Z}$-trees) to actions on arbitrary Λ-trees.

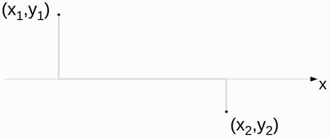
Examples for $\Lambda = \mathbb{R}$

$X = \mathbb{R}$ with usual metric.

A geometric realization of a simplicial tree.

$X = \mathbb{R}^2$ with metric d defined by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1| + |y_2| + |x_1 - x_2| & \text{if } x_1 \neq x_2 \\ |y_1 - y_2| & \text{if } x_1 = x_2 \end{cases}$$



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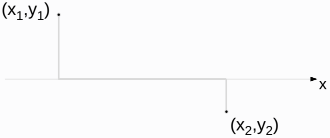

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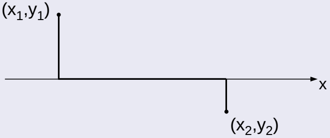

Finitely generated $\mathbb{R}$-free groups

Rips' Theorem [Rips, 1991 - not published]

A f.g. group acts freely on $\mathbb{R}$-tree if and only if it is a free product of surface groups (except for the non-orientable surfaces of genus 1,2, 3) and free abelian groups of finite rank.

Gaboriau, Levitt, Paulin (1994) gave a complete proof of Rips' Theorem.

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The Fundamental Problem

The following is a principal step in the Alperin-Bass' program:

Open Problem [Rips, Bass]

Describe finitely generated groups acting freely on Λ-trees.

Here "describe" means "describe in the standard group-theoretic terms".

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Theorem (H.Bass, 1991)

A finitely generated $(\Lambda \oplus \mathbb{Z})$-free group is the fundamental group of a finite graph of groups with properties:

- vertex groups are Λ-free,
- edge groups are maximal abelian (in the vertex groups),
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Theorem [Guirardel, 2003]

A f.g. freely indecomposable $\mathbb{R}^n$-free group is isomorphic to the fundamental group of a finite graph of groups, where each vertex group is f.g. $\mathbb{R}^{n-1}$-free, and each edge group is cyclic.

However, the converse is not true.

Corollary A f.g. $\mathbb{R}^n$-free group is hyperbolic relative to abelian subgroups.

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Theorem [Kharlampovich, Miasnikov, 96]

Finitely generated fully residually free groups are $\mathbb{Z}^n$-free.

Theorem [Martino and Rourke, 2005]

Let G_1 and G_2 be $\mathbb{Z}^n$-free groups. Then the amalgamated product $G_1 *_C G_2$ is $\mathbb{Z}^m$-free for some $m \in \mathbb{N}$, provided C is cyclic and maximal abelian in both factors.

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Examples of $\mathbb{Z}^n$-free groups:

$\mathbb{R}$-free groups,

$\langle x_1, x_2, x_3 \mid x_1^2 x_2^2 x_3^2 = 1 \rangle$ is $\mathbb{Z}^2$-free (but is neither $\mathbb{R}$-free, nor fully residually free).

The latter answers Conjecture Q 3.4 on Bestwina's problem page in the negative: it is not true that any $\mathbb{R}^n$-free (or even $\mathbb{Z}^n$-free) group is fully residually free.

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Every finitely generated Λ-free group is $\mathbb{Z}^n$-free.

All known finitely generated Λ-free groups are $\mathbb{Z}^m$-free.

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Length functions

Length functions were introduced by **Lyndon (1963)**.

Let G be a group. A function $l : G \to \Lambda$ is called a length function on G if

(L1) $\forall\, g \in G : \; l(g) \geqslant 0$ and $l(1) = 0$,

(L2) $\forall\, g \in G : \; l(g) = l(g^{-1})$,

(L3) the triple $\{c(g,f), c(g,h), c(f,h)\}$ is isosceles for all $g, f, h \in G$, where $c(f,g)$ is the Gromov's product:

$$c(g,f) = \frac{1}{2}(l(g) + l(f) - l(g^{-1}f)).$$

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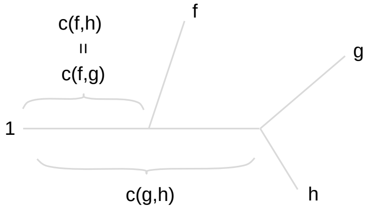
Examples of length functions

In a free group F, the function $f \to |f|$ is a $\mathbb{Z}$-valued length function.

For $f, g, h \in F$ we have



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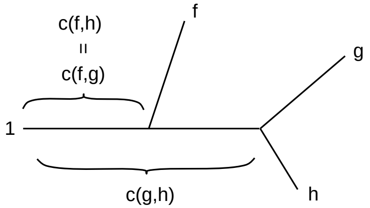

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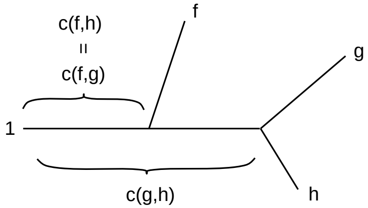

Free length functions

A length function $l : G \to \Lambda$ is free if $l(g^2) > l(g)$ for every non-trivial $g \in G$.

Chiswell's Theorem

Let (X, d) be a Λ-tree and $x \in X$. If G acts on (X, d) then $L(g) = d(x, gx)$ is a Lyndon length function on G.

Theorem [Chiswell]

Let $L : G \to \Lambda$ be a Lyndon length function on a group G. Then there exists a Λ-tree (X, d), $x \in X$, and an isometric action of G on X such that $L(g) = d(x, gx)$ for all $g \in G$.

Notice that $L(g) = d(x, gx)$ is free iff the action of G on X is free.

This gives another approach to free actions on Λ-trees.

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Infinite words

Let Λ be a discretely ordered abelian group with a minimal positive element 1_Λ and $X = \{x_i \mid i \in I\}$ be a set.

An **Λ-word** is a function

$$w : [1_\Lambda, \alpha] \to X^{\pm}, \quad \alpha \in \Lambda.$$

$|w| = \alpha$ is called the length of w.

w is **reduced** $\Longleftrightarrow$ no subwords xx^{-1}, $x^{-1}x$ $(x \in X)$.

$R(\Lambda, X) =$ the set of all reduced Λ-words.

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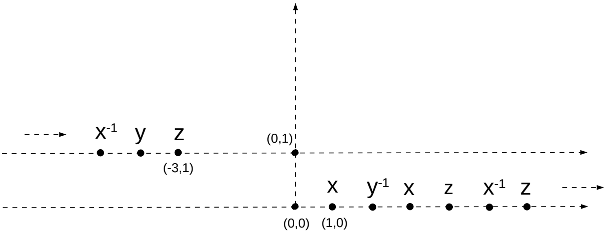
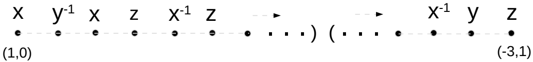
Example.

Let $X = \{x, y, z\},\ \Lambda = \mathbb{Z}^2$



In "linear" notation



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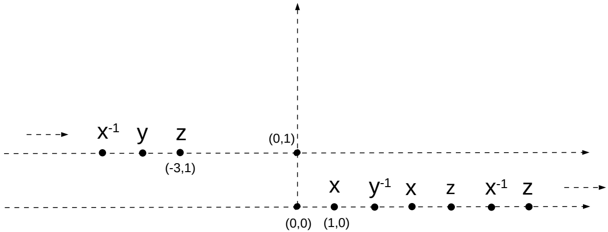

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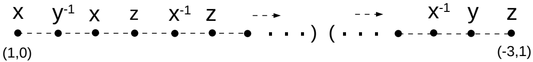
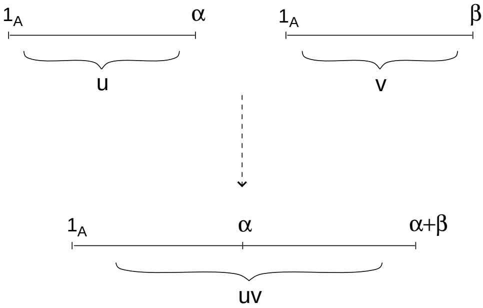
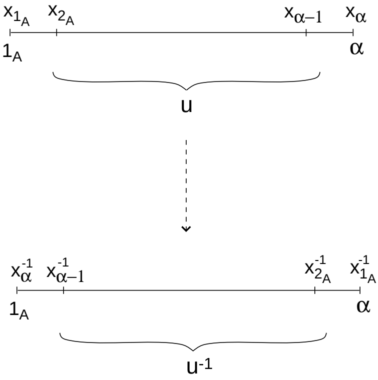


Concatenation of Λ-words:

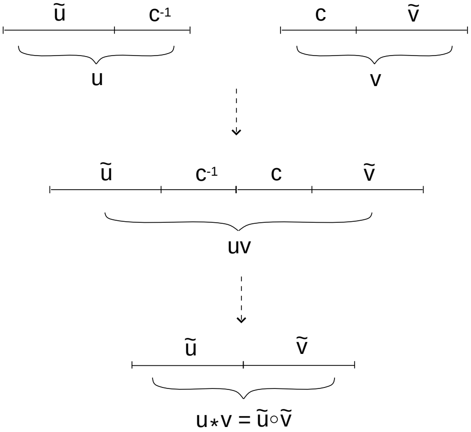


We write $u \circ v$ instead of uv in the case when uv is reduced.

Inversion of Λ-words:



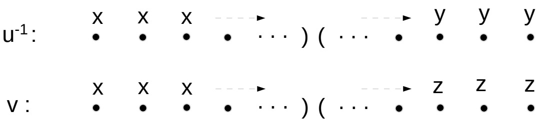
Multiplication of Λ-words:



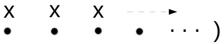
The partial group $R(\Lambda, X)$

The multiplication on $R(\Lambda, X)$ is partial, it is not everywhere defined!

Example. $u, v \in R(\mathbb{Z}^2, X)$



Hence, the common initial part of u^{-1} and v is



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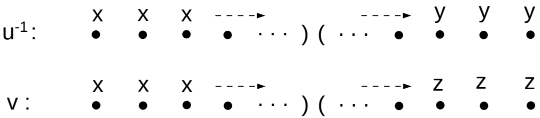
Example. $u, v \in R(\mathbb{Z}^2, X)$



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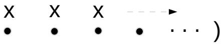

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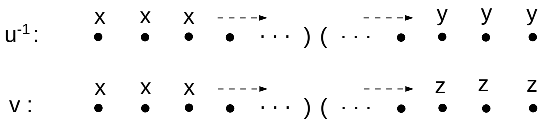

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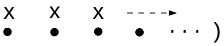

Cyclic decompositions

$v \in R(\Lambda, X)$ is **cyclically reduced** if $v(1_A)^{-1} \neq v(|v|)$.

$v \in R(\Lambda, X)$ admits a **cyclic decomposition** if

$$v = c^{-1} \circ u \circ c,$$

where $c, u \in R(A, \Lambda)$ and u is cyclically reduced.

Denote by $CDR(A, \Lambda)$ the set of all words from $R(\Lambda, X)$ which admit a cyclic decomposition.

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From Non-Archimedean words - to length functions

Theorem [Myasnikov-Remeslennikov-Serbin]

Let Λ be a discretely ordered abelian group and X a set. If G is a subgroup of $CDR(\Lambda, X)$ then the function $L_G : G \to \Lambda$, defined by $L_G(g) = |g|$, is a free Lyndon length function.

Corollary.

To show that a group G acts on a Λ-tree - embed G into $CDR(\Lambda, X)$.

Which Λ-free groups embed into $CDR(\Lambda, X)$?

All of them!

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From Length functions - to Non-Archimedean words

Theorem [Chiswell]

Let Λ be a discretely ordered abelian group. If $L : G \to \Lambda$ is a free Lyndon length function on a group G then there exists an embedding $\phi : G \to CDR(\Lambda, X)$ such that $|\phi(g)| = L(g)$ for every $g \in G$.

Corollary. Let Λ be an arbitrary ordered abelian group. If $L : G \to \Lambda$ is a free Lyndon length function on a group G then there exists a length preserving embedding $\phi : G \to CDR(\Lambda', X)$, where $\Lambda' = \mathbb{Z} \oplus \Lambda$ with the lex order.

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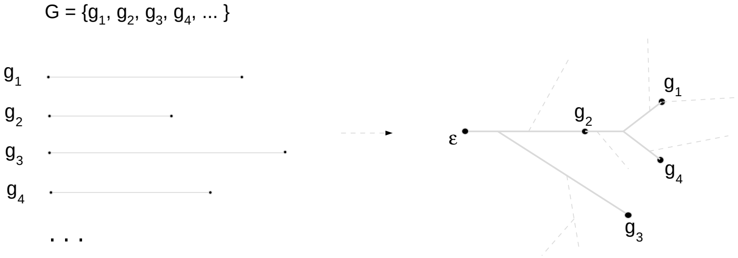
From Non-Archimedean words - to free actions

Infinite words $\Longrightarrow$ Length functions $\Longrightarrow$ Free actions

Shortcut

If $G \hookrightarrow CDR(\Lambda, X)$ then G acts by isometries on the canonical Λ-tree $\Gamma(G)$ labeled by letters from $X^{\pm}$.



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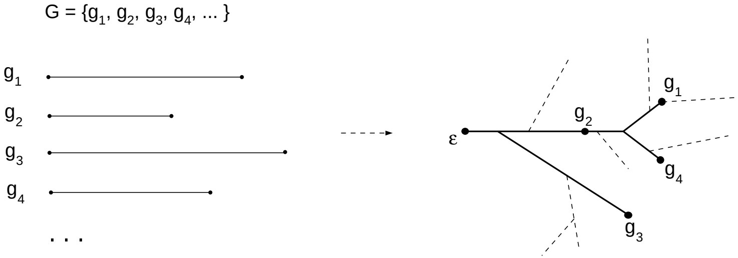

Complete free actions

Let $G \leq CDR(\Lambda, X)$ be a group of infinite words.

Complete subgroups

$G \leq CDR(\Lambda, X)$ is complete if G contains the common initial segment $c(g, h)$ for every pair of elements $g, h \in G$.

Complete length functions

A Lyndon length function $L : G \to \Lambda$ is complete (regular) if there exists a length preserving embedding $G \to CDR(\Lambda, X)$ onto a complete subgroup.

Remark One can define completeness (regularity) in terms of length functions or actions.

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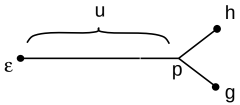
Branch points and completeness

A vertex $p \in \Gamma(G)$ is a branch point if it is the terminal endpoint of the common initial segment $u = com(g, h)$ of $g, h \in G$.



Completeness $\Longrightarrow$ all branch points are in one G-orbit of Γ

Theorem [Kharlampovich, Myasnikov, Serbin]

Every finitely generated $\mathbb{Z}^n$-free group is a subgroup of a finitely generated complete $\mathbb{Z}^k$-free group.

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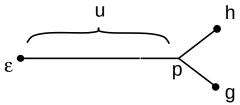

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Finitely presented complete Λ-free groups.

Theorem [Kharlampovich, Myasnikov, Serbin]

If G is f.p. and has a complete free length function in Λ, then G has an index two subgroup that can be represented as a union of a finite series of groups

$$G_1 < G_2 < \cdots < G_n = G,$$

where

1. G_1 is a free group,
2. G_{i+1} is obtained from G_i by finitely many HNN-extensions in which associated subgroups are maximal abelian and length-isomorphic.

Theorem [Kharlampovich, Myasnikov, Serbin]

Any f.p. group G with a complete free length function in an ordered abelian group Λ has an index two subgroup with free length function in $\mathbb{Z}^k$ ordered lexicographically for an appropriate $k \in \mathbb{N}$.

Elimination processes and free actions

Infinite branches of an elimination process correspond precisely to the standard types of free actions:
linear case $\Longleftrightarrow$ **thin (or Levitt)** type
the quadratic case $\Longleftrightarrow$ **surface type (or interval exchange)**,
periodic structures $\Longleftrightarrow$ **toral (or axial)** type.

Bestvina-Feighn's elimination process

A powerful variation of the Makanin-Razborov's process for $\mathbb{R}$-actions.
Can be viewed as an asymptotic (limit) version of MR process.
Much simpler in applications but ineffective.

KM elimination process for $\mathbb{Z}^n$ actions

To solve equations in fully residually free groups we designed a variation of the elimination process for $\mathbb{Z}^n$ actions.

It **effectively** describes solution sets of finite systems of equations in $\mathbb{Z}^n$-groups in terms of **Triangular quasi-quadratic systems** (as in the case of fully residually free groups).

Non-standard version of Rip's machine

Recently, **Kh., Myasnikov, and Serbin** designed an elimination process for arbitrary non-Archimedean actions, i.e, free actions on Λ-trees.

This can be viewed as a **non-Archimedean (non-standard)** discrete, effective version of the original MR process.

Sketch of the proof of the theorem about Λ-free f.p. groups

Let G have a regular free length function in Λ.

Fix an embedding of G into $CDR(\Lambda, X)$ and construct a cancellation tree for each relation of G.

Sketch of the proof

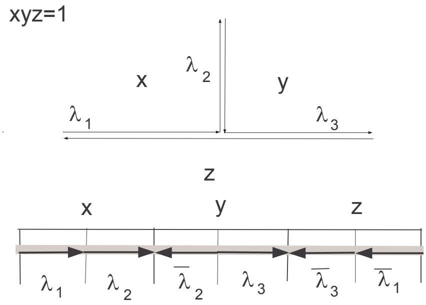
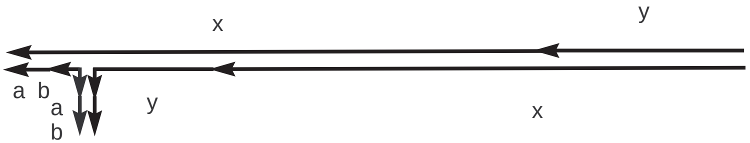
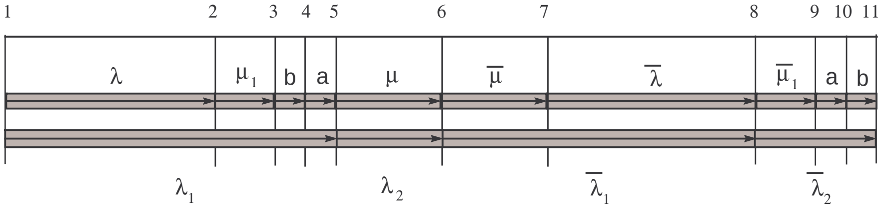


Figure: From the cancellation tree for the relation $xyz = 1$ to the generalized equation $(x = \lambda_1 \circ \lambda_2,\ y = \lambda_2^{-1} \circ \lambda_3,\ z = \lambda_3^{-1} \circ \lambda_1^{-1})$.

$[x,y][b,a]=1$





Sketch of the proof

Infinite branches of an elimination process correspond to abelian splittings of G:

linear case $\Longleftrightarrow$ **splitting as a free product.**

the quadratic case $\Longleftrightarrow$ **QH-subgroup**,

periodic structures $\Longleftrightarrow$ **abelian** vertex group or splitting as an HNN with abelian edge group.

After obtaining a splitting we apply EP to the vertex groups. We build the Delzant-Potyagailo hierarchy.

The General Case

Conjecture

Every finitely generated Λ-free group is a subgroup of a finitely generated complete Λ-free group.

If true then everything reduces to the complete case.

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Every finitely generated complete Λ-free group is finitely presented.

If true then by Bass-Serre theory every f.g. Λ-free group is finitely presented, and the general case follows.

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