Digraph groups and their applications

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Graph Groups

- Let $\Gamma = (V, E)$ be a finite simplicial graph.
- The graph group G_Γ is defined by the presentation

$$G_\Gamma = \langle V \mid vw = wv \text{ if } v \text{ and } w \text{ are connected by an edge in } \Gamma \rangle.$$

- Known also under variety of names:
- free partially commutative groups,
- right-angled Artin groups,
- semifree groups,
- traces groups.

The two extremes:

- $\Gamma-$ empty graph
- G_Γ is a free group F_V with a free generating set V.
- $\Gamma-$ complete graph.
- $G_\Gamma \simeq \mathbb{Z}V$ - a free abelian group with a basis V.
- In general, G_Γ can be thought of as interpolating between these two extremes.

Graph Group as a Functor

- $\Gamma \longrightarrow G_\Gamma$ - a covariant functor from the category of finite graphs to the category of groups.
- It takes full embeddings to monomorphisms.

Graph Groups in Robotics

- Let $CS_N(\Gamma)$ be the configuration space of N distinct points on a graph Γ.
- $CS_N(\Gamma) =$
  $\{\text{points in } \Gamma \times \cdots \times \Gamma \text{ with pairwise distinct coordinates}\}$.
- **Conjecture.** (Ghrist, 1999) For any finite graph Γ the fundamental group $\pi_1 CS_N(\Gamma)$ is a graph group.
- **Example.**

$$\pi_1 CS_4(\Gamma_H) = F_{195} * (*_6(\mathbb{Z} \times \mathbb{Z})).$$

Digraph Groups

- Given a directed graph $\Delta = (V, A)$ on a finite set V we define a digraph group G_Δ as follows:
- The set of arcs A is a generating set of G_Δ.

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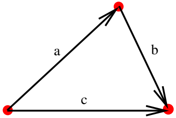
Defining Relations

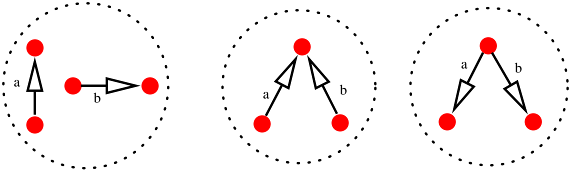
- Commutation relation

$$[a, b] = aba^{-1}b^{-1} = c$$

for each shortcut in Δ:

- 

- Commutativity relation for all a, b of the types:

- 

Digraph Group of a Ring

- Given a digraph Δ and a ring R (associative, with 1) we define a digraph group $G_\Delta(R)$ as follows:
- Generators are the symbols $a(r), (a \in A, r \in R)$.
- Relation $a(r+s) = a(r)a(s), (a \in A, r, s \in R)$.
- Commutation relation

$$[a(r), b(s)] = c(rs), \quad (a, b \in A, r, s \in R)$$

  for each shortcut (a, b, c) in Δ.
- Commutativity relation for all a, b of the types as above.

Connection with graph groups

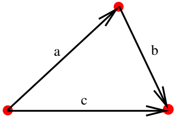
- If Δ does not contain shortcuts then G_Δ is a graph group.

Questions

- 1. *Is every graph group a digraph group?*
  2. *How large is the class of digraph groups?*

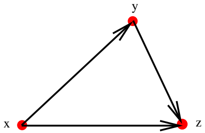
- Digraph groups are:
- finitely generated,
- finitely related,
- maybe nontrivial, except for the empty digraph.

Example - Heizenberg group

- Δ is a shortcut:
- 

- $$G_\Delta \simeq \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$$
  - the Heizenberg group.
- $$a \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, c \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Transitivity

- A digraph Δ is transitive when it has the property that any 2-path xyz is a shortcut i.e. xz is an arc.
- 

- In a transitive digraph every successor of every vertex can be reached in one step.
- Every digraph Δ has a transitive closure, that is, a least transitive digraph Δ_t, containing Δ.

Symmetric-oriented decomposition

- A digraph $\Delta = (V, A)$ is called symmetric if $xy \in A$ implies $yx \in A$.
- The opposite property is that of being an oriented digraph - this is a digraph with no cycle of length two.
- Every digraph Δ is a union of uniquely defined symmetric subgraph Δ_s and oriented subgraph Δ_o:

$$\Delta = \Delta_s \cup \Delta_o.$$

- If Δ were transitive then the components would be transitive too.
- A transitive symmetric digraph is a disjoint union of complete symmetric digraphs.

Levi decomposition of G_Δ

Theorem

Let $\Delta = \Delta_s \cup \Delta_o$ be a symmetric-oriented decomposition of a transitive digraph Δ. Then $G_{\Delta_s}, G_{\Delta_o}$ naturally embed into G_Δ. G_{Δ_o} is a nilpotent normal subgroup in G_Δ and G_Δ is a semi-direct product $G_\Delta = G_{\Delta_s} \ltimes G_{\Delta_o}$.

Question

Does the Levi-Malcev theorem hold true for transitive digraph groups? That is, are all the complements of G_{Δ_o} conjugated?

Steinberg groups

- Let R be a ring (associative with 1).
- For $n \geq 3$ define the Steinberg group $St_n(R)$ in terms of generators and relations (designed to imitate the behavior of the elementary matrices).
- Generators are the symbols $x_{ij}(r)$, with i, j a pair of distinct integers between 1 and n and $r \in R$.
- Steinberg relations:

1. $x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$,
2. $$[x_{ij}(r), x_{k\ell}(s)] = \begin{cases} 1 & \text{if} \quad i \neq l, j \neq k \\ x_{il}(rs) & \text{if} \quad i \neq l, j = k \\ x_{kj}(-rs) & \text{if} \quad i = l, j \neq k \end{cases}$$

Structure: transitive symmetric case

Theorem

Let Δ be a complete digraph on a set $V = \{1, \ldots, n\}$. Then:

1. *For $n = 1$ the group G_Δ is trivial;*
2. *For $n = 2$ the group G_Δ is a free group on generators $12, 21$;*
3. *For $n \geq 3$ there is an isomorphism $\pi : G_\Delta \simeq St_n(\mathbb{Z})$ that sends the arc ij to $x_{ij}(1), 1 \leq i \neq j \leq n$.*

Theorem

A transitive symmetric digraph group G_Δ is a direct product of finitely many groups chosen from the following list (repetitions allowed): F_2, $St_n(\mathbb{Z}), n \geq 3$.

Structure: transitive oriented case

Theorem

1. *Every transitive oriented digraph group G_Δ is nilpotent,*
2. *The nilpotence degree of G_Δ equals the longest length of a path in Δ,*
3. *The Malcev rank of G_Δ equals $|A|$.*

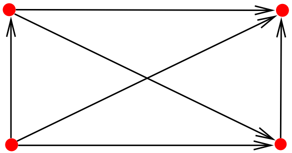


Figure: Nilpotent digraph group of rank 6 and degree 3

Linear representation

Questions

1. *Does a digraph group admit a faithful (finite dimensional) linear representation?*
2. *Does a transitive digraph group admit a faithful (finite dimensional) linear representation?*

- Let Δ be a digraph. Associated to an arc uv is a $V \times V$-matrix (=transvection)

$$t_{uv} = e + e_{uv},$$

  where e_{uv} is a $V \times V$-matrix which has 1 on (u, v)-slot and 0 elsewhere.
- A linear representation $\tau : G_\Delta \to SL(\mathbb{Z}V)$ is given by

$$a = uv \mapsto t_{uv}, \forall a \in A.$$

Theorem

(J. Milnor, Introduction to Algebraic K-theory thm. 10.1) For $n \geq 3$ the group $St_n(\mathbb{Z})$ is a central extension of the form

$$C_n \rightarrowtail St_n(\mathbb{Z}) \twoheadrightarrow SL_n(\mathbb{Z}),$$

where C_2 is a cyclic group of order 2, generated by the symbol $(x_{12}x_{21}^{-1}x_{12})^4$. For $n = 2$ there is an exact sequence

$$R \rightarrowtail F_2 \twoheadrightarrow SL_2(\mathbb{Z}),$$

where

$$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, y \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and R is the normal closure of $xy^{-1}xyx^{-1}y, (xy^{-1}x)^4$.

Theorem

If Δ is a transitive oriented digraph (or equivalently transitive acyclic digraph) then τ is faithful.

Proof.

Sinks, sources, induction. □

Corollary

$\mathrm{Ker}(\tau)$ is finite iff each 2-cycle in Δ is contained in a complete symmetric graph on 3 vertices.

Levi decomposition of $\mathrm{Im}(\tau)$

- Denote $\tau(G_\Delta) = T_\Delta$.

Theorem

Let $\Delta = \Delta_s \cup \Delta_o$ be a symmetric-oriented decomposition of a transitive digraph Δ. Then $T_{\Delta_s}, T_{\Delta_o}$ naturally embed into T_Δ. T_{Δ_o} is a normal nilpotent subgroup in T_Δ isomorphic to G_{Δ_o} and $T_\Delta = T_{\Delta_s} \ltimes T_{\Delta_o}$.

Theorem

For any directed graph Δ the group $\tau(G_\Delta)$ is an arithmetic subgroup in $GL(\mathbb{R}V)$.

-

Structure of T_Δ

- $M_\Delta(\mathbb{R})$ = all matrices $a = (a_{uv})$ with the set of indices $V \times V$, such that

$$a_{uv} = 0 \text{ if } u \neq v \text{ and } uv \notin \Delta. \tag{1}$$

  M_Δ is a subring of the full matrix ring $Mat_n(\mathbb{R})$.

- Then we define

$$H_\Delta(\mathbb{R}) = M_\Delta(\mathbb{R})^\times. \tag{2}$$

- $H_\Delta(\mathbb{R})$ is a linear algebraic $\mathbb{Q}$-defined group and $H_\Delta(\mathbb{Z})$ is an arithmetic subgroup in $H_\Delta(\mathbb{R})$.

Theorem

T_Γ is of finite index in $H_\Delta(\mathbb{Z})$.

Proof.

Three steps.

1. $H_\Delta(\mathbb{R})$ is solvable
2. $H_\Delta(\mathbb{R})$ is reductive.
3. For the general case we use the Levi decomposition of $G_\Delta(\mathbb{R})$.

□

Group Theory versus Digraph Theory

Theorem

Let Δ be a finite digraph. The group G_Δ is finite if and only if Δ is empty.

Theorem

For a finite transitive digraph Δ the following conditions are equivalent:

1. *Δ is acyclic (i.e. does not contain a directed cycle),*
2. *G_Δ is virtually nilpotent,*
3. *G_Δ is virtually solvable.*

Group Theory versus Digraph Theory

Theorem

If Δ is a finite transitive digraph then G_Δ is virtually abelian if and only if Δ is acyclic and has no paths of length 2.

Theorem

Every transitive digraph group is either virtually nilpotent or contains a copy of F_2.

Transvection automorphisms of graph groups

- Let G_Γ be a graph group.
- A transvection occurs for each pair of distinct vertices u, v such that $\mathrm{lk}\,(u) \subseteq \mathrm{st}\,(v)$. In this case the transvection τ_{uv} sends u to uv and fixes the other generators.
- A star digraph Δ of Γ has the same set of vertices V and $uv \in V \times V$ is an arc in Δ if and only if $u \neq v$ and $\mathrm{st}\,(u) \subseteq \mathrm{st}\,(v)$ in Γ.
- A (more refined) link digraph Δ of Γ has the same set of vertices V and $uv \in V \times V$ is an arc in Δ if and only if $u \neq v$ and $\mathrm{lk}\,(u) \subseteq \mathrm{st}\,(v)$ in Γ.

Example: link digraph

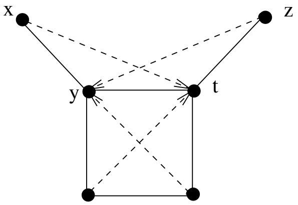


Figure: Dashed is a link digraph of a solid graph

A star digraph group acts on a graph group

- Let Δ be a star digraph of a graph Γ.
- A representation $\alpha : G_\Delta \mapsto SAut(G_\Gamma)$ is given by

$$a = uv \mapsto \tau_{uv}, \quad \forall a \in A.$$

- The subgroup $S = \alpha(G_\Delta)$ was introduced by
  Duncan & Kazachkov & Remeslennikov (2007).
- They show that S is arithmetic.
- Motivation: why S is linear?
- Raffiniert ist der Herrgot, aber boshaft ist er nicht.
- Subtle is the Lord, but malicious He is not.
- A guess – S maps injectively under abelianization.

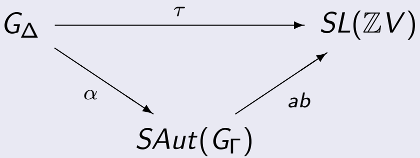
Theorem

DKR-group S is mapped injectively under abelianization!

Naturality of α and τ

Lemma

The following diagram is commutative



Detailed version of the diagram:

$$\begin{array}{ccc} G_{\Delta_s} \ltimes G_{\Delta_o} & \xrightarrow{\quad\tau\quad} & T_{\Delta_s} \ltimes T_{\Delta_o} \\ & {\scriptstyle \alpha}\searrow \qquad \nearrow{\scriptstyle ab} & \\ & SAut(G_\Gamma) \supseteq S_{\Delta_s} \ltimes S_{\Delta_o} & \end{array}$$

- τ is injective on G_{Δ_o},
- ab is injective on S_{Δ_o},
- ab is injective on S_{Δ_s} by an abstract Lemma:

Lemma

Let G be a group generated by a set V and assume that V is "independent" modulo the commutator subgroup i.e. a natural homomorphism $G/G' \to \mathbb{Z}V$ is bijective. Suppose $V = \cup V_i$ is a partition of V into commutative subsets and denote $G_i = gp\langle V_i \rangle$. Then the stabilizer S in $Aut(G)$ of the set $\{G_i\}$ is mapped injectively under abelianization.

Lemma

Let G_Γ be a graph group. Suppose $V = \cup V_i$ is a partition of V into commutative subsets and denote $G_i = gp\langle V_i \rangle$. Then the stabilizer S in $Aut(G)$ of the set $\{G_i\}$ is mapped injectively under abelianization.

Applications to the property T for automorphism group of a graph group

Theorem

There is a graph Γ with $|V| = 6$ such that $Aut(G_\Gamma)$ has positive first Betti number, i.e. it has a subgroup which maps onto $\mathbb{Z}$. Consequently $Aut(G_\Gamma)$ does not have property T.

Applications to the abelianization for graph groups

Theorem

For any finite graph Γ the image of the abelianization map

$$\alpha : Aut\,(G_\Gamma) \to GL\,(\mathbb{Z}V) \tag{3}$$

is an arithmetic subgroup of $GL\,(\mathbb{R}V)$.

- Last problem - prove something interesting about digraph groups!

Thanks!