A Geometric Zero-one Law

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Outline

Zero-One Laws: I will discuss a **geometric Zero-One Law** for infinite graphs. The principal examples are Cayley graphs of infinite groups. A a "geometric" version of Zero-One Law states that for any first-order sentence ϕ of graph theory either ϕ or its negation $\neg\phi$ holds almost surely on finite subgraphs of a graph X.

Reduction to graphs: One can approach an arbitrary algebraic structure X via its Gaifman graph $[X]$ that brings "geometry" into play.

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Coarse First-order Logic: similar to the coarse geometry one can consider a coarse logic. For example the coarse elementary theory of a structure X consists of all sentences that almost surely true in X.

Random substructures: One can define random substructures of X similar to the Erdos' model. It turns out they are also analogs of the Rado universal graphs. We show that random substructures give models of the coarse first-order theory of X: they are almost surely elementarily equivalent to each other, though not necessarily isomorphic.

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The classical Zero-One Law: a first order sentence is either almost true or almost false for finite structures of a fixed purely relational language $\mathcal{L}$.

More precisely: if for each n we select randomly and uniformly an n-element relational structure, then the probability that a selected structure satisfies the fixed sentence goes to either 1 or 0 as n approaches infinity.

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Two types of Zero-One Laws

More precisely:

Labeled Zero-One Law

In this case a structure is chosen at random with respect to the uniform distribution on all structures with universe $\{1, 2, \ldots, n\}$.

Compare with the Erdos' model of a "random graph" on n vertices.

Unlabeled Zero-One Law

In this case one chooses an isomorphism class of structures uniformly at random from the set of isomorphism classes of structures with universe of size n.

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Extensions of Zero-One Law

The labeled version has received the greater share of attention.

It holds for models of parametric axioms, graphs for example, i.e., undirected graphs without loops.

There are many extensions of the Zero-One Law to different logics and different probability distributions.

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Replacing operations by predicates

One can replace an operation $f : A \times \ldots \times A \to A$ on a set A by a $(n+1)$-ary relation R_f

$$R_f(a_1, \ldots, a_n, b) = 1 \iff f(a_1, \ldots, a_n, b) = b.$$

This way an algebraic structure X can be replaced by a "similar" relational structure X'.

There are differences: every finite subset of X forms a substructure.

For instance, finite substructures of a group G are finite **partial** subgroups.

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Gaifman graphs

Gaifman graph $[X]$ of a structure X in the language $\mathcal{L}$:

vertices of $[X]$ = elements of X

two distinct vertices x, y are connected by an undirected edge in $[X]$ if there is a relation $R(z_1, \ldots, z_\ell)$ in $\mathcal{L}$ with $x = z_i$ and $y = z_j$ for some i, j.

For graphs $X = [X]$ and the notions above are the standard ones.

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Metric notions

Distance $d(x, y)$ = the length of the shortest path from x to y in $[X]$ or ∞ if there is no such path.

The ball $B_n(x)$ = the substructure of X supported by the elements at distance at most n from a given $x \in X$.

If $S \subset X$ then $B_n(S) = \cup_{x \in S} B_n(x)$.

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Connectivity and degrees

A structure X is connected if the graph $[X]$ is connected.

If all vertices of $[X]$ have finite degree then X is locally finite.

If the vertex degrees in $[X]$ are uniformly bounded then X has bounded degree.

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Almost true sentences

Definition

Let X be an infinite, connected, locally finite structure. A sentence ϕ is almost surely true for finite substructures of X if for every $x \in X$ the fraction of substructures of $B_n(x)$ for which ϕ is true approaches 1 as n approaches infinity. Likewise ϕ is almost surely false if that fraction approaches 0 as n approaches infinity.

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Coarse universal theory

The universal theory of X tells something about finite substructures of X':

Model theory theorem

Two algebraic structures X and Y are universally equivalent iff X' and Y' have the same finite submodels.

Under proper circumstances, limit groups of G = models of the universal theory of G.

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The main technical property

Substructures Y and Z of X are *disjoint* if $[Y] \cap [Z] = \emptyset$ and there are no edges between them.

Duplication

A structure X has the duplicate substructure property if for every finite substructure there is a disjoint isomorphic substructure.

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The Main Theorem

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Let X be infinite connected and of bounded degree. If X has the duplicate substructure property then any sentence is either almost true or almost false for finite substructures of X.

Structures where The Main Theorem applies

Cayley Diagrams

The Cayley diagram of a finitely generated infinite group. Here the language consists of one binary relation for each generator.

Vertex-transitive graphs

An infinite connected graph of finite degree whose automorphism group is transitive on vertices. For example, the underlying graph obtained from a Cayley diagram (but there are non-Cayley examples)

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The Cayley diagram of a free finitely generated monoid.

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The full binary tree, i.e., the tree with one vertex of degree two and all others of degree three. More generally the full k-ary tree for $k \geq 1$.

An infinite connected locally finite and finite dimensional simplicial complex whose automorphism group is transitive on zero-simplices. There is one $n+1$-ary relation for simplices of each dimension.

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Distinguishing from the classical case

A unary forest is a directed acyclic graph such that each vertex has at most one incoming edge and at most one outgoing edge.

Let X be the disjoint union of finite unary forests with edges labeled by 0 and 1. Here the language consists of two binary relations, one for each edge label.

Theorem

X obeys the geometric zero-one law but does not obey either the labeled or unlabeled law.

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Erdos random graphs

A random graph is obtained by starting with a set of n vertices and adding edges between them at random.

Most commonly studied is the Erdos-Renyi model, denoted in which every possible edge occurs independently with probability p.

If instead we start with an infinite set of vertices, and again let every possible edge occur independently with probability p, then we get an infinite random graph, or Erdos graph.

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- A Rado graph is unique (up to isomorphism).
- A Rado graph contains all finite and countably infinite graphs as induced subgraphs.

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Random subgraphs of Γ

Let Γ be a connected graph of bounded degree.

For a fixed p, $0 < p < 1$, one may imagine a process of generating a random subgraph by deleting each vertex of Γ with probability $1 - p$.

The resulting subgraph consists of all remaining vertices and all edges between them in Γ.

A random subgraph of Γ is a subgraph that have a non-zero probability to occur in the process.

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Elementarily equivalence

Suppose the graph Γ satisfies the duplication property.

Let T_Γ be the set of all sentences that almost surely true on the finite subgraphs of Γ.

Theorem

With probability 1 a random subgraph of Γ is a model of T_Γ. In particular, almost all random subgraphs of Γ are elementarily equivalent.

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The measure

A more precise definition of random substructures of X is obtained by first defining a measure on cones.

For each pair, S, T, of disjoint finite subsets of elements of X, the corresponding cone consists of all subsets of elements which include S and avoid T.

The measure of this cone is defined to be $p^{|S|}q^{|T|}$, where $|S|$ and $|T|$ are the cardinalities of S and T respectively, and $q = 1 - p$.

By a well known theorem of Carathéodory the measure on cones extends uniquely to a probability measure, μ, on the σ–algebra generated by the cones.

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Percolation

Let Γ be a Cayley graph of a finitely generated group G (relative to a given finite generated set).

Let $0 \leq p \leq 1$. The *Bernoulli bond percolation* on a graph Γ is defined as our generating procedure for random subgraphs of Γ only here one deletes edges with the probability $1 - p$, not the vertices.

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Clusters = connected components of the resulted random subgraph R.

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Critical values

$\theta(p) =$ the probability that the identity element 1 of G belongs to an infinite cluster of R.

$\zeta(p) =$ the probability that there exists exactly one infinite cluster in R.

Critical values:

$$p_c(G,S) = \sup\{p \mid \theta(p) = 0\}, \quad p_u(G,S) = \inf\{p \mid \zeta(p) = 1\}$$

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Percolation behavior of Cayley graphs

Theorem [Benjamini and Schramm]

For every Cayley graph Γ there are constants $0 \leq p_c \leq p_u \leq 1$ such that:

- for $p \in (0, p_c)$ the graph Γ_p does not have clusters.
- for $p \in (p_c, p_u)$ the graph Γ_p has infinitely many clusters.
- for $p \in (p_u, 1)$ the graph Γ_p has precisely one cluster.

Surprisingly, these numbers tell a lot about the structure of the initial group G!

For most groups all these intervals are not empty, $0 < p_c < p_u < 1$.

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Critical Percolation p_c

Conjecture

If $\Gamma = \Gamma(G, X)$ is the Cayley graph of an infinite finitely generated group G, which is not a finite extension of $\mathbb{Z}$, then $p_c(\Gamma) < 1$.

Lyons (1994): for Cayley graphs of exponential volume growth.

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Cheeger's constant of a graph Γ

$$h(\Gamma) = \inf_{S \subset_{fin} \Gamma} \frac{|\partial S|}{|S|}$$

where ∂S is the boundary of S, i.e., the set of vertices in $\Gamma \setminus S$ that have a neighbor in S.

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Similarity of clusters

Let $p \in (p_c, p_u)$ so there are infinitely many clusters in Γ_p.

It is known that the clusters in general are not isomorphic. How different they could be?

Theorem [Gilman, Gurevich, Myasnikov]

The following holds:

- Any two clusters of Γ are logically equivalent, - they satisfy precisely the same formulas of the first-order logic.
- If the Word Problem in a group G is decidable then there is an algorithm to decide if a given formula is true in the cluster of a Cayley graph of G or not.

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Two references on percolation

- I. Benjamini and O. Schramm, Percolation beyond Z^d, many questions and a few answers. Electron. Comm. Probab., 1 (no. 8) , 71-82 (electronic), 1996.
- BS2 I. Benjamini and O. Schramm, Recent progress on percolation beyond Z^d, http://research.microsoft.com/~schramm/pyondrep/.

Back to the future: from groups - to other structures

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It seems probable that the critical constants again describe something essential for these structures.

Perhaps, some coarse geometry also exists in this case.

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Classes of finite structures

Hypothesis

Let $\mathcal{F}$ be a class of finite structures satisfying the following conditions.

1. $\mathcal{F}$ is closed under taking substructures.
2. $\mathcal{F}$ has bounded degree.
3. If F_1 and F_2 are (not necessarily distinct) elements of $\mathcal{F}$, then there exists an element of $\mathcal{F}$ isomorphic to the disjoint union $F_1 \cup F_2$.
4. $\mathcal{F}$ is *pseudo-connected* in the sense that for every $F \in \mathcal{F}$ there is an embedding of F into a connected member of $\mathcal{F}$.

Random structures as generic models

Let $\mathcal{F}$ be a class of finite structures satisfying the Hypothesis, and let S be the disjoint union of all members of $\mathcal{F}$. We have:

1. There is an infinite structure X, called an *ambient structure* for $\mathcal{F}$, such that X satisfies the hypotheses of the Main Theorem, and the finite substructures of X are the same as the elements of $\mathcal{F}$ up to isomorphism.
2. Let X be any ambient structure for $\mathcal{F}$. Then an arbitrary first-order sentence is almost surely true for finite structures of X if and only if it holds in S. Consequently all ambient structures give the same same zero-one law on $\mathcal{F}$.