1. [10]	2. [10]	3. [10]	4. [10]	5. [10]	6. [10]	7. [10]	Total
8. [10]	9. [10]	10. [10]	11. [10]	12. [10]	13. [10]	14. [10]	Total
					Final Grade:		

MA116 - Final (May 13, 2010)

Name: Solutions *Pipeline Username:*

Circle your lecture: A-Kazmierczak (9:00) B-Kazmierczak (10:00) C-Dubovski (11:00) D-Dubovski (12:00) E-Wang(12:00)

Pledge and Sign:

Show all work. Answers without supporting work will not receive credit. You may not use a calculator, cell phone, or computer while taking this exam.

1. [10 pts] Evaluate the following limits:

(a) $\lim_{x\to 0} \frac{1-\cos(4x)}{\sin^2 x}$

(b) $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

a) $\lim_{x\to 0} \frac{1-\cos(4x)}{\sin^2 x} \overset{LH}{=} \lim_{x\to 0} \frac{4\sin(4x)}{2\sin x\cos x} \overset{LH}{=} \lim_{x\to 0} \frac{16\cos(4x)}{2\cos^2 x - 2\sin^2 x}$

$= \boxed{8}$

- or -

$\lim_{x\to 0} \frac{1-\cos(4x)}{\sin^2 x} \overset{LH}{=} \lim_{x\to 0} \frac{4\sin(4x)}{2\sin x\cos x} = \lim_{x\to 0} \frac{4\sin(4x)}{\sin(2x)}$

$\overset{LH}{=} \lim_{x\to 0} \frac{16\cos(4x)}{2\cos(2x)} = \boxed{8}$

b) $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = \lim_{x\to\infty} e^{\ln(1+\frac{2}{x})^x}$

consider

$\lim_{x\to\infty} \ln\left(1+\frac{2}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right) =$

$= \lim_{x\to\infty} \frac{\ln(1+\frac{2}{x})}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{1+2/x}\cdot(-2x^{-2})}{-x^{-2}} =$

$= \lim_{x\to\infty} \frac{2}{1+2/x} = 2$

So, $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = \lim_{x\to\infty} e^{\ln(1+\frac{2}{x})^x} = e^{\lim_{x\to\infty}\ln(1+\frac{2}{x})^x}$

$= \boxed{e^2}$

2. [10 pts] Determine whether the following improper integrals are convergent or divergent. If convergent, evaluate it. If divergent, give a reason.

a) $\int_1^5 \frac{1}{(x-1)^2}dx$

b) $\int_e^\infty \frac{1}{x(\ln x)^2}dx$

a) $\int_1^5 \frac{1}{(x-1)^2}\,dx = \lim_{t\to 1^+} \int_t^5 \frac{1}{(x-1)^2}\,dx = \lim_{t\to 1^+} -(x-1)^{-1}\Big|_t^5 =$

$= -\frac{1}{4} + \lim_{t\to 1^+} \frac{1}{t-1}$

But $\lim_{t\to 1^+} \frac{1}{t-1} = \infty$. Therefore, divergent.

b) $\int_e^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^2}\,dx =$

$= \lim_{t\to\infty} -\frac{1}{\ln x}\Big|_e^t = \lim_{t\to\infty}\left(-\frac{1}{\ln t}\right) + 1 = 1$

Convergent to 1.

3. [10 pts] Determine whether the given series is convergent or divergent, and justify your answer.

a) $\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$

b) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

c) $\sum_{n=1}^{\infty} \pi^n e^{-n}$

b) Note that $\left|\frac{\sin n}{n^2}\right| < \frac{1}{n^2}$ for $n \geq 1$

so by the comparison test

$\sum \frac{\sin n}{n^2}$ converges since $\sum \frac{1}{n^2}$ converges

c) $\sum_{n=1}^{\infty} \pi^n e^{-n} = \sum_{n=1}^{\infty} \left(\frac{\pi}{e}\right)^n$ but $\frac{\pi}{e} > 1$

and the series is geometric so it diverges.

a) Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!(n+1)!}{(2n+2)!}}{\frac{n!\,n!}{2n!}} \right| = \lim \left| \frac{(n+1)!(n+1)!(2n)!}{(2n+2)!\,n!\,n!} \right| =$$

$$\lim \left| \frac{(n+1)^2 (2n)!}{(2n+2)!} \right| = \lim \left| \frac{(n+1)^2}{(2n+2)(2n+1)} \right| = \lim \left| \frac{n^2+2n+1}{4n^2+6n+2} \right|$$

$= \frac{1}{4} < 1$ so it converges

4. [10 pts] Find all x for which the following series converges.

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$$

We use the ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{(x+2)^{n+1}}{\sqrt{n+1}}}{\frac{(x+2)^n}{\sqrt{n}}} \right| = \lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{(x+2)^n} \frac{\sqrt{n}}{\sqrt{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x+2)\sqrt{n}}{\sqrt{n+1}} \right| = |(x+2)| \lim_{n\to\infty} \left| \frac{\sqrt{n}}{\sqrt{n+1}} \right|$$

$|(x+2) \cdot 1| < 1$ so $|x+2| < 1 \Rightarrow$

$-\cancel{3 < x < -1}$ $\quad -1 < x+2 < 1 \Rightarrow -3 < x < -1$

Check end pts

$x=-3$ $\quad \sum_{n=1}^{\infty} \frac{(-3+2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by alternating series test

$x=-1$ $\quad \sum_{n=1}^{\infty} \frac{(-1+2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n}}$ diverges p series $p = 1/2$

So $[-3, 1)$ interval of convergence

5. [10 pts] Determine whether the following geometric series converge or diverge. If the series converges, determine the sum of the series.

a) $3+\frac{6}{5}+\frac{12}{25}+\frac{24}{125}+\dots$ $= 3\left(1+\frac{2}{5}+\left(\frac{2}{5}\right)^2+\left(\frac{2}{5}\right)^3+\cdots+\left(\frac{2}{5}\right)^n+\cdots\right)$ infinite geometric series

b) $\sum_{n=0}^{\infty}(-3)^{n+1}5^{-n}$

$= 3\left[\frac{1}{1-\frac{2}{5}}\right]$ ratio $= \frac{2}{5} < 1 \Rightarrow$ Series converges

$= 3 \times \frac{5}{3}$

$= 5$ #

b) $= \sum(-3)^n \cdot (-3)(5^{-n})$

$= (-3)\sum_{n=0}^{\infty}\left(-\frac{3}{5}\right)^n$ $r = -\frac{3}{5}$ $-1 < r < 1$

$\Rightarrow$ Series converges

$= -3 \times \frac{1}{1-(-\frac{3}{5})}$

$= -3 \times \frac{5}{8}$

$= \frac{-15}{8}$ #

6. [10 pts] Find the Taylor series expansion for the following. Give your answer in compact form (i.e. find the general term).

(a) $f(x) = \cos x$, centered at $a = \pi/2$.

(b) $f(x) = x^2 e^{2x}$, centered at $a = 0$ (i.e. Maclaurin series).

			term
(a)	$f(x) = \cos x$	$f(\frac{\pi}{2}) = 0$	0
	$f'(x) = -\sin x$	$f'(\frac{\pi}{2}) = -1$	$-(x-\frac{\pi}{2})$
	$f''(x) = -\cos x$	$f''(\frac{\pi}{2}) = 0$	0
	$f'''(x) = \sin x$	$f'''(\frac{\pi}{2}) = 1$	$\frac{1}{3!}(x-\frac{\pi}{2})^3$
	$f^{(4)}(x) = \cos x$	$f^{(4)}(\frac{\pi}{2}) = 0$	0
	$\vdots$	$\vdots$	

$$\cos x = 0 - (x-\tfrac{\pi}{2}) + 0 + \frac{1}{3!}(x-\tfrac{\pi}{2})^3 + 0 - \frac{1}{5!}(x-\tfrac{\pi}{2})^5 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(2n+1)!} (x-\tfrac{\pi}{2})^{2n+1} \quad \#$$

$$\left(\text{or} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n-1)!} (x-\tfrac{\pi}{2})^{2n-1} \right)$$

b) Maclaurin series for $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges for all real numbers x

$$\therefore e^{2x} = \sum \frac{1}{n!} (2x)^n$$

$$\therefore x^2 e^{2x} = x^2 \sum \frac{1}{n!} (2x)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} 2^n \cdot x^{n+2} \quad \#$$

7. [10 pts]

(a) Find the angle between the vectors $\langle 4,2,2\rangle$ and $\langle 1,2,-1\rangle$.

(b) For what values of b are the vectors $\langle -6,b,2\rangle$ and $\langle 1,-2b,4b\rangle$ orthogonal?

a) $a = \langle 4,2,2\rangle \quad b = \langle 1,2,-1\rangle$

$a\cdot b = |a||b|\cos\theta \Rightarrow \cos\theta = \dfrac{a\cdot b}{|a||b|} = \dfrac{4(1)+2(2)+2(-1)}{\sqrt{24}\sqrt{6}} = \dfrac{6}{2(6)} = \dfrac{1}{2}$

$\theta = \cos^{-1}(\tfrac{1}{2}) = \boxed{\tfrac{\pi}{3}}$

-or-

$|a\times b| = |a||b|\sin\theta \Rightarrow \sin\theta = \dfrac{|a\times b|}{|a||b|}$

$a\times b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 2 & 2 \\ 1 & 2 & -1 \end{vmatrix} = [2(-1)-2(2)]\vec{i} - [4(-1)-2(1)]\vec{j} + [4(2)-2(1)]\vec{k}$

$= -6\vec{i} + 6\vec{j} + 6\vec{k}$; $|a\times b| = \sqrt{(-6)^2+6^2+6^2} = \sqrt{108} = 6\sqrt{3}$

$\sin\theta = \dfrac{6\sqrt{3}}{\sqrt{24}\sqrt{6}} = \dfrac{6\sqrt{3}}{2(6)} = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = \sin^{-1}(\tfrac{\sqrt{3}}{2}) = \boxed{\tfrac{\pi}{3}}$

b) $a' = \langle -6,b,2\rangle \quad b' = \langle 1,-2b,4b\rangle$ orthogonal $\Rightarrow a'\cdot b' = 0$

$a'\cdot b' = -6(1) + b(-2b) + 2(4b) = -6-2b^2+8b = 0 \Rightarrow b^2-4b+3=0$

$\Rightarrow (b-3)(b-1) = 0 \qquad \boxed{b = 1,3}$

8. [10 pts] Find the equation of the tangent plane for the surface $xe^y\cos z - z = 1$ at the point $(1,0,0)$

Equation of a Tangent Plane (3 variables):

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

$F(x,y,z) = xe^y\cos z - z - 1 = 0$

$F_x = e^y\cos z \qquad F_x(1,0,0) = e^0\cos(0) = 1$

$F_y = xe^y\cos z \qquad F_y(1,0,0) = 1e^0\cos(0) = 1$

$F_z = -xe^y\sin z - 1 \qquad F_z(1,0,0) = -1e^0\sin(0)-1 = 0-1 = -1$

Eq $\Rightarrow$ $(x-1)+(y-0)-(z-0)=0 \Rightarrow \boxed{x+y-z=1}$

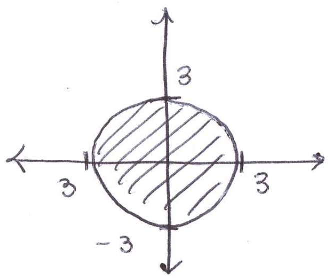
-or- Equation with 2 variables: $\dfrac{dz}{dx}(x-x_0) + \dfrac{dz}{dy}(y-y_0) = z-z_0$

where $\dfrac{dz}{dx} = -\dfrac{F_x}{F_z} = -\dfrac{e^y\cos z}{-xe^y\sin z-1}$, Plug in $(1,0,0)$: $\dfrac{dz}{dx} = \dfrac{-1}{-1} = 1$

$\dfrac{dz}{dy} = -\dfrac{F_y}{F_z} = -\dfrac{xe^y\cos z}{-xe^y\sin z-1}$, Plug in $(1,0,0)$: $\dfrac{dz}{dy} = \dfrac{-1}{-1} = 1$

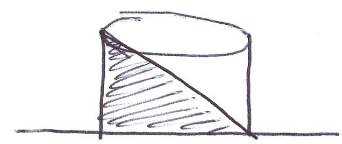
So, $(x-1)+(y-0) = z-0 \Rightarrow \boxed{x+y-z=1}$

9. [10 pts] Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $x + z = 3$. [*Suggestion*: Use polar coordinates]



$x^2+y^2=9$

Circle centered at $(0,0)$, radius 3



$R=\{(r,\theta)\mid 0\le r\le 3,\ 0\le\theta\le 2\pi\}$

Now $x+z=3 \Rightarrow z=3-x$

$$V=\iint (3-x)\,dA = \int_0^{2\pi}\int_0^3 (3-r\cos\theta)\,r\,dr\,d\theta =$$

$$= \int_0^{2\pi}\int_0^3 (3r-r^2\cos\theta)\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{3}{2}r^2-\tfrac{1}{3}r^3\cos\theta\right]_{r=0}^{r=3} d\theta =$$

$$= \int_0^{2\pi}\left[\left(\tfrac{3}{2}(3)^2-\tfrac{1}{3}(3)^3\cos\theta\right)-\left(\tfrac{3}{2}(0)^2-\tfrac{1}{3}(0)^3\cos\theta\right)\right]d\theta =$$

$$= \int_0^{2\pi}\left(\tfrac{27}{2}-9\cos\theta\right)d\theta = \left[\tfrac{27}{2}\theta-9\sin\theta\right]_0^{2\pi} =$$

$$= \left[\left(\tfrac{27}{2}(2\pi)-9\sin(2\pi)\right)-\left(\tfrac{27}{2}(0)-9\sin(0)\right)\right] =$$

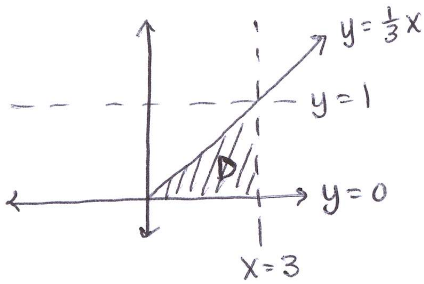
$$= \boxed{27\pi}$$

10. [10 pts] Consider the integral $\int_0^1 \int_{3y}^3 e^{x^2}\, dx dy$.

a) Sketch the domain of integration.
b) Write the equivalent integral by reversing the order of integration.
c) Evaluate the integral.

a) $D = \{(x,y) \mid 0 \le y \le 1,\ 3y \le x \le 3\}$ Type 2

$3y = x \Rightarrow y = \frac{1}{3}x$



b) $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le \frac{1}{3}x\}$ Type 1

$$\int_0^3 \int_0^{1/3 x} e^{x^2}\, dy dx$$

c) $$\int_0^3 \int_0^{1/3 x} e^{x^2}\, dy dx = \int_0^3 \Big[y e^{x^2} \Big]_{y=0}^{y=\frac{1}{3}x} dx =$$

$$= \int_0^3 \Big[\tfrac{1}{3} x e^{x^2} - 0(e^{x^2}) \Big] dx = \frac{1}{3} \int_0^3 x e^{x^2} dx = \frac{1}{3} \int_0^9 e^u \frac{du}{2}$$

$u = x^2$
$du = 2x dx$
$\frac{du}{2} = x dx$

$x = 0 \Rightarrow u = 0$
$x = 3 \Rightarrow u = 9$

$$= \frac{1}{6} \int_0^9 e^u du = \frac{1}{6} e^u \Big|_0^9$$

$$= \frac{1}{6}(e^9 - e^0)$$

$$= \boxed{\frac{1}{6}(e^9 - 1)}$$

11. [10 pts] Find the parametric equations for the line of intersection of the following two planes:

$$2x+3y-4z=-9$$
$$3x-4y+2z=12$$

[**Hint:** Since the intersecting line lies in each of the planes, its direction must be perpendicular to the normal of each plane.]

$2x+3y-4z=-9$ $\qquad$ $3x-4y+2z=12$

$\vec{n}_1 = \langle 2,3,-4\rangle$ $\qquad$ $\vec{n}_2 = \langle 3,-4,2\rangle$

$$\vec{n} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -4 \\ 3 & -4 & 2 \end{vmatrix} = [3(2)-(-4)(-4)]\vec{i} - [2(2)-(-4)(3)]\vec{j} +$$

$$+[2(-4)-3(3)]\vec{k} = -10\vec{i} + 16\vec{j} - 17\vec{k}$$

$$\vec{n} = \langle a,b,c\rangle = \langle -10, 16, -17\rangle$$

Let $z=0$: $\quad 2x+3y-4(0)=-9$, $\; 3x-4y+2(0)=12$ $\Rightarrow$ $2x+3y=-9$, $\; 3x-4y=12$ $\Rightarrow$

$\Rightarrow$ $8x+12y=-36$
$9x-12y=36$

$17x=0$
$x=0$,

$2(0)+3y=-9$
$3y=-9$
$y=-3$

$(0,-3,0)$ point of intersection on the line

Parametric Equations:

$x = x_0 + at$, $\quad y = y_0 + bt$, $\quad z = z_0 + ct$

$x = -10t$ $\quad y = -3+16t$ $\quad z = -17t$

12. [10 pts] Find the absolute maximum and minimum values of $f(x,y) = x^2 + xy + y^2$ on the region $\{(x,y) \mid x^2 + y^2 \le 1\}$. [**Hint:** Remember this means you must find the values of f at the critical points of f in $x^2 + y^2 < 1$ and then also find extreme values of f on the constraint $x^2 + y^2 = 1$]

using Lagrange Multipliers:

$f(x,y) = x^2 + xy + y^2 \Rightarrow \nabla f = \langle 2x+y,\ x+2y \rangle = \langle 0,0 \rangle$

Critical Points: $x = 0,\ y = 0 \qquad (0,0)$

Now, $(0,0)$ lies on the region $x^2 + y^2 < 1$. On the boundary $g(x,y) = x^2 + y^2 = 1$,

$$\lambda \nabla g = \langle 2\lambda x,\ 2\lambda y \rangle$$

Now, $\nabla f = \lambda \nabla g \Rightarrow 2x+y = 2\lambda x$ and $x + 2y = 2\lambda y$

So, $\lambda = \dfrac{2x+y}{2x}$ and $\lambda = \dfrac{x+2y}{2y}$

If $\lambda = 0$ then $x = 0,\ y = 0$, so $(0,0)$ [same as above]

If $\lambda \neq 0 \Rightarrow \dfrac{2x+y}{2x} = \dfrac{x+2y}{2y} \Rightarrow 2y(2x+y) = 2x(x+2y)$

$\Rightarrow 4xy + 2y^2 = 2x^2 + 4xy \Rightarrow 2y^2 = 2x^2 \Rightarrow x^2 = y^2$.

using the constraint, $x^2 + y^2 = 1 \Rightarrow x^2 + x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow$
$x = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$ and hence $y = \pm \frac{\sqrt{2}}{2}$.

Thus, $f(0,0) = 0$ and $f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \frac{3}{2}$ and
$f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{1}{2}$.

Absolute minimum - $f(0,0) = 0$

Absolute maximum - $f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \frac{3}{2}$

13. [10 pts]

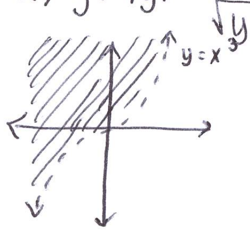
a) Find and sketch the domain of $g(x,y)=\dfrac{1}{\sqrt{y-x^3}}$. What is its range?

b) Consider the function $z=f(x,y)=y-x^3$. Sketch the level curves for $z=-1$, $z=0$, and $z=1$.

c) Sketch the curve traced out by the position vector $\vec{r}(t)=t\hat{\imath}+(t^2+1)\hat{\jmath}$

a) $g(x,y)=\dfrac{1}{\sqrt{y-x^3}}$ Domain: $y-x^3>0 \Rightarrow y>x^3$ $D(g)=\{(x,y) \mid y>x^3\}$

Range: $R(g)=(0,\infty)$



c) $\vec{r}(t)=t\vec{\imath}+(t^2+1)\vec{j}$

$x=t$

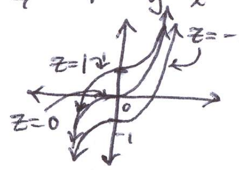
$y=t^2+1$ $\Rightarrow$ $y=x^2+1$

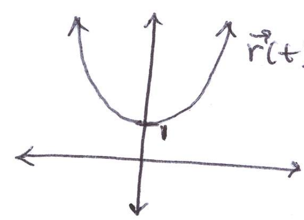
b) $z=f(x,y)=y-x^3$

$z=-1 \Rightarrow -1=y-x^3 \Rightarrow y=x^3-1$

$z=0 \Rightarrow 0=y-x^3 \Rightarrow y=x^3$

$z=1 \Rightarrow 1=y-x^3 \Rightarrow y=x^3+1$





14. [10 pts] If $f(x,y,z)=x\sin yz$ find the following:

a) The gradient of f

b) The directional derivative of f at $(1,3,0)$ in the direction of $\vec{v}=\hat{\imath}+2\hat{\jmath}-\hat{k}$

a) $\nabla f=\langle f_x, f_y, f_z\rangle=\langle \sin(yz),\ xz\cos(yz),\ xy\cos(yz)\rangle$

b) $D_u f(1,3,0)=\nabla f(1,3,0)\cdot\vec{u}$

$\nabla f(1,3,0)=\langle \sin(3\cdot 0),\ 1(0)\cos(3\cdot 0),\ 1(3)\cos(3\cdot 0)\rangle=\langle 0,0,3\rangle$

$\vec{u}=\dfrac{\vec{v}}{|\vec{v}|}=\dfrac{\langle 1,2,-1\rangle}{\sqrt{1^2+2^2+(-1)^2}}=\left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right\rangle$

$D_u f(1,3,0)=\langle 0,0,3\rangle\cdot\left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right\rangle=0\left(\frac{1}{\sqrt{6}}\right)+0\left(\frac{2}{\sqrt{6}}\right)+3\left(\frac{-1}{\sqrt{6}}\right)$

$=-\dfrac{3}{\sqrt{6}}$