Lecture 4. (X_i, d_i), $o_i \in X_i$,

ω - ultrafilter.

$$\Pi_b(X_i, d_i) = \{(x_i),\ \underline{d}((x_i),(o_i))<\infty\}$$

$$d((x_i),(y_i)) = \lim^\omega (d_i(x_i, y_i))$$

$$\lim^\omega (X_i, d_i) = \Pi_b(X_i, d_i)/\sim$$

$$(x_i) \sim (y_i) \iff d = 0$$

$$X_i = X/p_i = (X, d/p_i)$$

$$\lim^\omega (X_i, d/p_i) = Con^\omega(X, (p_i), (o_i))$$

---

1) X - homogeneous $\Rightarrow Con^\omega$ is (o_i)-independent

$G \curvearrowright X$, $\Pi_b G/\omega \curvearrowright Con^\omega$

2) $X \underset{q.i.}{\sim} Y$

$$\varphi : X \to Y$$

$$\frac{1}{\lambda}\frac{d}{p_i} - \frac{C}{p_i} \le \frac{d(\varphi(x),\varphi(y))}{p_i} \le \lambda\frac{d(x,y)}{p_i} + \frac{C}{p_i}$$

$$Con^\omega(X) \underset{\lambda\text{-b.L.}}{\sim} Con^\omega(Y)$$

3) $Con^\omega(X)$ - complete

4) $Y_i \subseteq X$; $\lim^\omega (Y_i)$

$Y_i = q_i[0, l_i]$

$\Rightarrow$ $\underline{\lim Y_i}$ is a path $q_i : [0, \lim^\omega l_i/p_i]$

$(q_i(t_i)) \overset{def}{=} q(\lim^\omega t_i)$

limit of geod is geod.

---

5) $Con^\omega(X) \supseteq \bigcirc \sim \bigcirc = \lim \bigcirc_i \subseteq X$

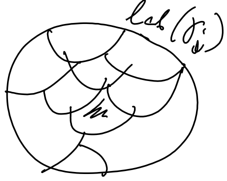
Th (Gromov) $G = \langle X \rangle$. If all a.c. of G are simply connected, then

a) G is f.p.

b) G has polynomial Dehn function

Pf Suppose G is not f.p. $\Rightarrow$

$\exists\ r_i$ $\ell_{sb}(r_i)$ is not a corollary of shorter words (relations)



$p_i = |\gamma_i|$

$Con^\omega(G, (p_i)) = \lim^\omega (X/p_i)$

γ_i has length 1 in X/p_i)

$\gamma = \lim^\omega \gamma_i$, γ is a loop in Cone of length 1.

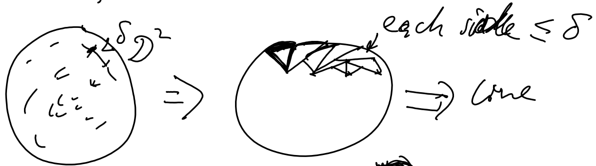
$\exists$ cont. map $\varphi: D^2 \to$ Cone

$S^1 \to \gamma$

D^2 is compact

φ is uniformly continuous

Take $\varepsilon = \frac{1}{5}$, find δ



In the Cayley graph

γ_i perimeter of each triangle is $\frac{3}{5}|\gamma_i| + o(|\gamma_i|)$
$\le \frac{4}{5}|\gamma_i|$

lab(γ_i) is a relation which follows from relations of length $\le \frac{4}{5}|\gamma_i|$

ω-a.s. | $o^\omega(p_i)$

Poly Dehn function

$f(x)$ is poly. $\Rightarrow$

$f(2x) < K f(x)$

Suppose there is no polynomial s.t.
$Dehn(n) \le f(n)$

$\Rightarrow \forall K \ \exists \gamma_K$ $\boxed{Area(\gamma_K) \geq K \, Area(\mu)}$

$|.| < |\gamma_K|$

$|\mu| < \dfrac{|\gamma_K|}{2}$

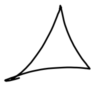
Do the same as above, subdivide D^2 into 10^{70} of little triangles $\Rightarrow$
$K = 10^{70}$ is a contradiction

Hyperbolic groups

Def  ← geod / is in δ-neigh. of

$\Rightarrow$ δ-hyperbolic

Divide the dist by 2 $\Rightarrow$

 $\Rightarrow$ / is in $\delta/2$-neigh. of

$\lim^\omega (X/p_i)$ – complete 0-hyp. geod space

The a.c. of every hyp. space is an $\mathbb{R}$-tree

If X is not q.i. to $\mathbb{Z}$
then a.c. are all isometric to the universal $\mathbb{R}$-tree – $\mathbb{R}$-tree of degree continuum at every vertex

(Erschler + Polterovich).

Th A group is hyperbolic iff all a.c. are trees.

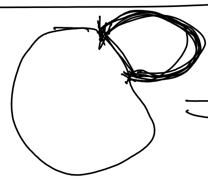
Corollaries

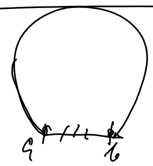
1) Approximation (Delzant)
2) Prove all unproved statements in Gromov's paper "Hyp. groups"

Th Every hyp. group has Dehn presentation and a linear Dehn function

A presentation $G = \langle X \mid R \rangle$ is Dehn if.
Every word $W = 1$ in G has more than

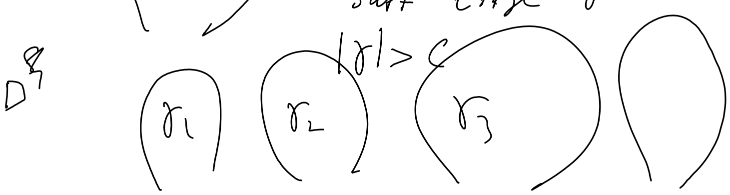
a length of a relator

Dehn.  $\Rightarrow$ linear isop. inequality (Dehn function)

Bump  $|a-b| \le \frac{1}{3}|\gamma|$

Theorem Every bump γ has a subbump of length $\ge \frac{|\gamma|}{3} > 0$

 (true for all suff large γ $|\gamma| > C$

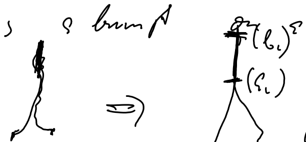


length $(\gamma_i) = p_i$

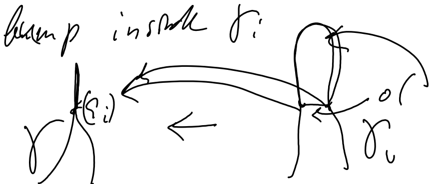
$X/p_i \Longrightarrow Con(X/p_i)$

$\lim \gamma_i$ is a bumb γ in Cone

γ is a bump a tree

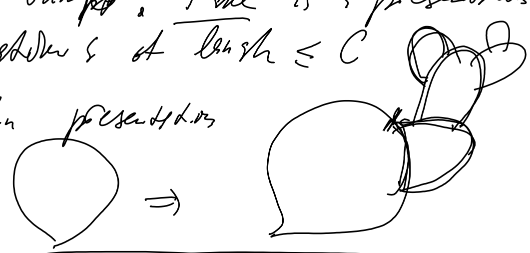
 $[(a_i),(b_i)] = \lim \delta_i$, $\delta_i \subseteq \gamma_i$

Then the subpath (a_i, b_i) is a bump inside γ_i



$\Longrightarrow$ Every loop of length $\ge C$ has a bump. Take as a presentation all relations of length $\le C$

$\Rightarrow$ Dehn presentation



G is hyp. $\Longleftrightarrow$ Dehn function is linear

There is a universal constant C

such that if the Dehn function is $\Theta(n) \leq C n^2 \Rightarrow$ group is hyperbolic.

No Dehn functions between n^1 and n^2

Lecture 5. Nilpotent groups. Quadratic Dehn functions

Th (Papasoglu) If As-function (of all a.c. of a group) is bounded by n^c, then the Dehn function of the group $\leq n^{c+\varepsilon}$ for every ε. $\forall \varepsilon$.

(That is for every ε there exists a constant C_ε so that the Dehn function $\leq C_\varepsilon n^{c+\varepsilon}$)

Drutu used this theorem to estimate Dehn functions of f.g. nilpotent groups

Def: N - nilpotent group $[x, y, z, \dots] = 1$, here $[x,y,z] = [[x,y],z]$, $[x,y] = x^{-1}y^{-1}xy$

Th. For every nilpotent group N every a.c. $\mathrm{Con}^\omega(N,(d_i))$ is a Lie group constructed as follows

$Tor(N)$ - all elements of finite order

$N/tor(N)$ - torsion-free -

Co-compact lattice of $\bar N$

To construct $\bar N$ add formal powers x^α, $\alpha \in \mathbb{R}$ for all elements x of N

The a.c. is the homogeneous Lie group constructed from $\bar N$

$\mathrm{Hom}(\bar N)$

$\bar N > [\bar N, \bar N] > \dots \quad 1$

$\bar N > c_1(\bar N) > c_2(\bar N) > \dots 1$

$\bar N / c_1(\bar N) \oplus \dots \oplus c_k(\bar N)/c_{k+1}(\bar N) \leftarrow$ This is the set $\mathrm{Hom}(\bar N)$

Operation $\rightarrow [g\, c_i(N), h\, c_j(N)] = [g,h]\, c_{i+j}(N)$

$N > c_1(N) > c_2(N) > \dots -$

$X_1 \qquad X_2$

$[x, y] =$

$x \in c_i/c_{i+1} \quad y \in c_j/c_{j+1}$

$[x,y] \in c_{i+j}$

$\mathrm{Hom}(\bar N)$ is homogeneous because for every $x \in c_i(H) \setminus c_{i+1}(H)$, $y \in c_j(H) \setminus c_{j+1}(H)$, $[x,y] \in c_{i+j} \setminus c_{i+j+1}$ or $[x,y] = 1$

Pittet: As for $\mathrm{Hom}(\bar N)$ is $\leq n^{k+1}$ where k is the class of nilpotency

$N > [N, N] > 1$

The metric is Carnot-Carathéodory

http://intlpress.com/JDG/archive/1985/21-1-35.pdf

horizontal $x_1, x_2, \dots, x_k \quad y_1, \dots, y_s \leftarrow$ vertical

$c = [a, b]$, $c^{n^2} = [a^n, b^n]$; to travel n^2 up one can instead travel $4n$ horizontally

$\Rightarrow$ Th (Drutu) Every nilp. class c group has Dehn function at most n^{c+1} (use Papasoglu)

Gersten + Riley: n^{c+1} (Also Gromov)

Free nilp. group of class c has Dehn function n^{c+1}

"Dehn function" n^{c+1}

$H_3 = \langle a, b, c \mid [a,b]=c,\ ca = ac,\ cb = bc \rangle$ ← Cubic Dehn function

$H_5 = \langle a, b, x, y, c \mid [a,b] = c = [x,y] = c,\ [c,a] = [c,b] = [c,x] = [c,y] = 1,\ \{a,b\} \sim \{x,y\} \text{ commute} \rangle$

Dehn function is quadratic (Allcock, Olshanskii–Sapir)

Baby example $\mathbb{Z}^2 = \langle a, b \mid [a,b] = 1 \rangle$

W

$aba^{-1}b^{-1}$

ab

ba

$W = 1$ if and only if by inserting relation and reducing we get an empty word. We can at every step replace ab by ba. The number of steps – Dehn function. How many steps needed – at most $|W|^2$: move first letter of w to the right until it meets a^{-1}, etc. So the Dehn function of $\mathbb{Z}^2$ is quadratic

$H_5/\langle c\rangle = \langle a, b, x, y\rangle$. Free Abelian

$W =_{H_5} 1 \Rightarrow W$ is a commutator word.

$\Rightarrow W = c^s,\ W =_{H_5} [a,b]^p [x,y]^q$

use commutativity H_5 $\{a,b\} \sim \{x,y\}$

$W = W'_{ab} W''_{xy} = W'_{ab} W''_{ab} =$

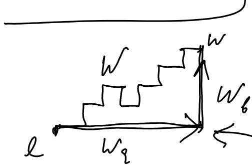
Th The Dehn function of H_5 is quadratic

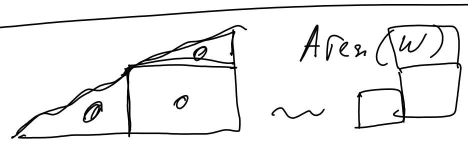
Lemma One can transform W_{ab} into W_{xy} in quadratic number of steps

It is enough to consider words in a, b

$W_W\ W W_b^{-1} W_a^{-1} = c^s$

Define the area of any word $w(a,b)$





Divide the triangle into 3 parts

Replace each part by a rectangle, replace a product of two rectangles by one rectangle.

Th $C_{10} = \langle a_1, b_1, \ldots, a_5, b_5 \mid [a_1,b_1][a_2,b_2]\cdots[a_5,b_5] = 1\rangle$

$C_{10} \times C_{10} / \{(u, u^{-1})\} = G_{10}$

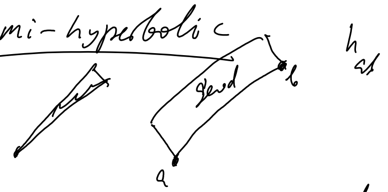
G_{10} has Dehn function at most $n^2 \log n$

Wenger: Dehn function of G_{10} is $\geq n^3$

Quadratic Dehn functions

Hyp. groups linear Dehn function

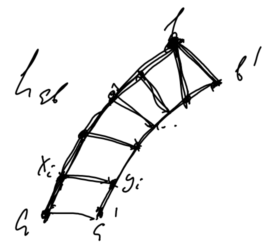
Semi-hyperbolic



$\forall a,b \ \exists$ geodesic h_{ab}

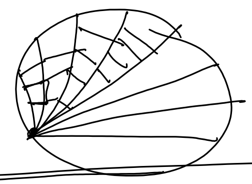
(i) $h_{ga,gb} = g\, h_{ab}$

(ii)



$$\mathrm{dist}(x_i, y_i) < C\,\big(\{\mathrm{dist}(a,a'), \mathrm{dist}(b,b')\}\big) + C_1$$

Statement: Every semihyp. group has quadratic Dehn function



Th The following groups have quadr. Dehn functions

1) **Druta** Polycyclic groups.

$\mathbb{Q} \subseteq K$ – totally real.

$\mathcal{O}(K) = \mathbb{Z}^n$ | $\mathbb{Z}^n \rtimes \mathbb{Z}^d$

$U(K) = \mathbb{Z}^d$

has Dehn function n^2

2) **Cornulier – Tessera** $\exists$ nonabelian group with quadr Dehn function $\supset BS(1,2)$.

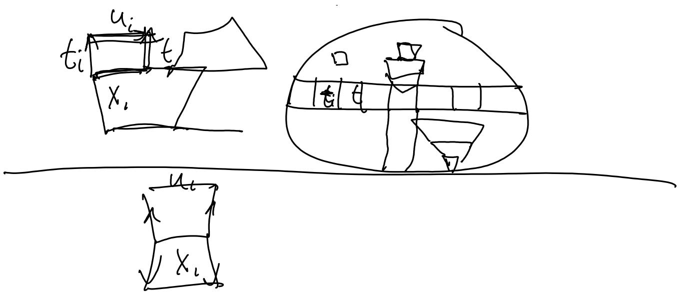
3) $F_k \rtimes \mathbb{Z}$ (Bridson + Groves)

– quadr Dehn function

4)

$F_k \rtimes \mathbb{Z}$ $\langle x_1, \dots, x_k, t$

$\underline{t^{-1} x_i t} = u_i$



Lecture 6. Conjugacy problem and Dehn functions.

Q-Th Suppose G - f.p. group, Dehn function is quadratic. Then the conjugacy problem is decidable.

i) Conjugacy problem

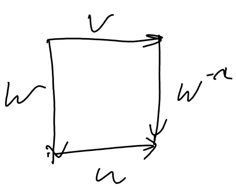
$G = \langle X \mid R \rangle \mid u, v \in (X \cup X^{-1})$

Question: $\exists ?\, w : w u w^{-1} = v$ in G

Can we bound the length of w in terms of $|u|$, $|v|$

Iff there is no effective bound on $|w|$, the problem is undecidable

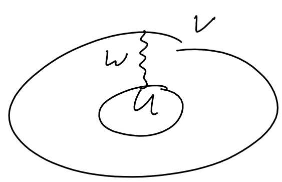
If $w u w^{-1} = v$ is true in G, then there is a vK diagram



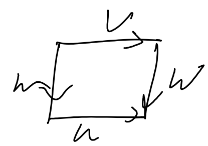
$\Longrightarrow$

– Schupp diagram

Conversely

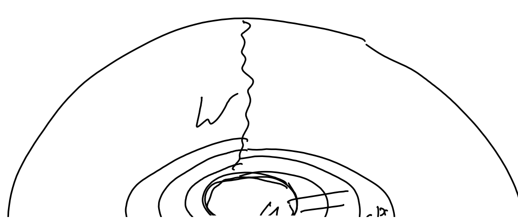


Cut along w



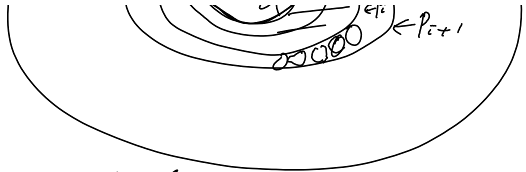
Suppose that the conj problem is not decidable. Need to prove that the Dehn function $> C n^2 \log n$

Consider



$x = |w| \gg |u|, |v|$

Suppose p_i



$P_0 = u$

is constructed

$P_i \cap V = \emptyset$

Take a union M_i of all cells that have common vertex with P_i, not between P_i, P_0

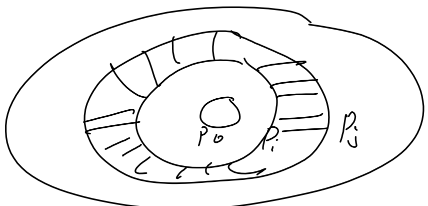
$\partial(M_i) \underset{def}{=} P_i \cup P_{i+1}$

Every vertex of P_i is at distance $O(1)$ from a vertex of P_{i+1}

M_i, M_j $(i \neq j)$ do not intersect

$|x| \sim$ # of layers M_i

Suppose P_i, P_j have the same label.



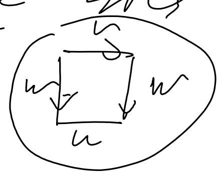
← remove

get a smaller diagram, a contradiction

At least half of x layers P_i have length $> \log_c x$

At least half of M_i has at least $O(\log x)$ cells.

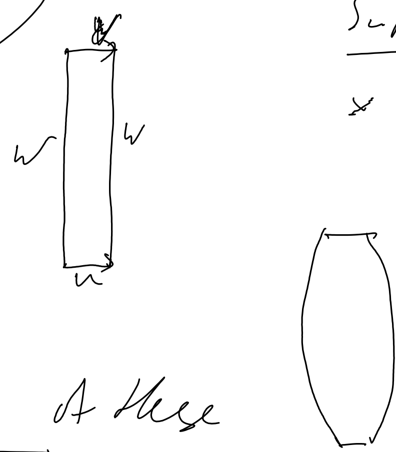
The area of the annulus and the area of



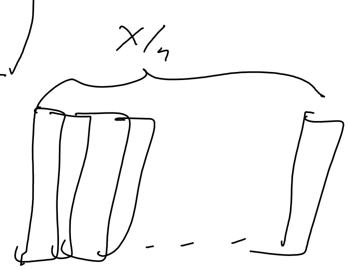
is $x \log x$ at least

Suppose $\max(|u|, |v|) = n$

$x >> n$



Glue $\frac{x}{n}$ of these diagrams side-by-side



perimeter $\leq 4x$

area $\frac{x^2 \log x}{n} \sim x^2 \log x$

The hole - there may be another diagram with the same boundary label, smaller area

Multiple HNN extensions of free groups $\to$ works

Examples of groups with quadr. Dehn functions

a) Semihyperbolic

b) Automatic groups

c) $F = \langle x_0, x_1 \mid \text{two def. rel.} \rangle$

$\langle x_0, x_1, \dots \mid x_i x_j x_i^{-1} = x_{j+1},\ i<j \rangle$

$D_F(n) \sim n^2$ (Guba)

Conjugacy is decidable (Guba - Sapir)

d) $SL_n(\mathbb{Z})$, $n \geq 5$. Dehn function $\sim n^2$ (Young)

$SL_n(\mathbb{R}) / SO_n(\mathbb{R})$ - symmetric space

$n=2$ $= \mathbb{H}^2$

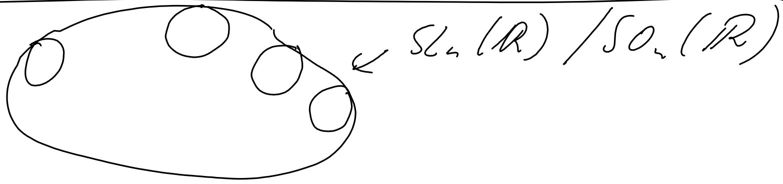
CAT(0) (semi-hyperbolic)

$SL_n(\mathbb{R})/SO_n(\mathbb{R})$ has quadratic area function

$SL_n(\mathbb{Z}) \curvearrowright$ on the Symmetric Space

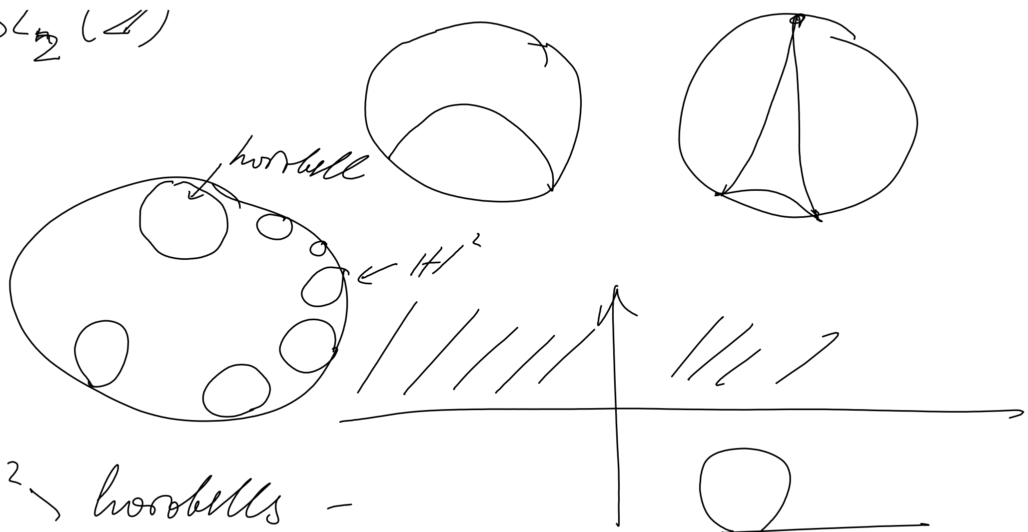
If the action was co-compact, the Dehn function of $SL_n(\mathbb{Z})$ would be quadratic

The action is not co-compact



$SL_n(\mathbb{Z})$

$SL_2(\mathbb{Z})$



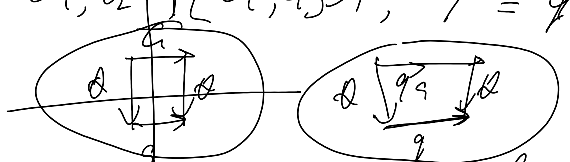
$\mathbb{H}^2 \setminus$ horoballs –
space on which $SL_2(\mathbb{Z})$ acts co-compactly.
This space has quadratic area function
in $n \geq 5 \Rightarrow$ D. f. of $SL_n(\mathbb{Z})$ is quadratic.
Conjugacy is decidable by Grunewald & Sarkisyan

e) $G \triangleright N \cong F_k$, G/N is cyclic
$\Rightarrow$ Dehn function is quadratic (Bridson + Groves)
Conjugacy is decidable (By Ventura, Bogopolski)

Th (Olshanskii, S.) $\exists$ f.p. multiple HNN of free group with undecidable C.P. and Dehn function $n^2 \log n$

S-machines

$$\langle a, q, \theta_1, \theta_2 \mid [\theta_i, a] = 1,\ q^{\theta_i} = qa \rangle$$



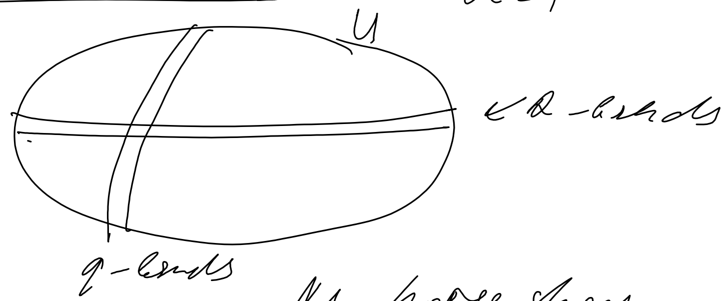
Normal subgroup $\langle a, q \rangle$ free, factor $\langle \theta_1, \theta_2 \rangle$ free
free-by-free

Dehn function?

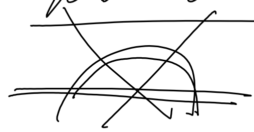
Theorem D. f. is cubic

Pf. 1) At most cubic $\qquad u = 1$

Pf. 1) At most cubic $u = 1$



No horse shoes



cells $\theta \; q \; \theta$ is $\sim |u|^2/4$

The cells $\theta \, \square \, \theta$ form q-bands

$\Rightarrow$ the number of q-bands is at most the number of q-cells plus the $|u|$

The length of an q-band is at most # of θ-bands $\Rightarrow \leq |u|/2$

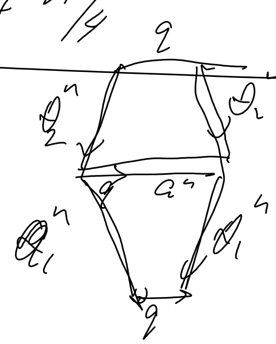
So the number of q-cells is at most $|u|^2 \cdot |u| / 8$

at most cubic

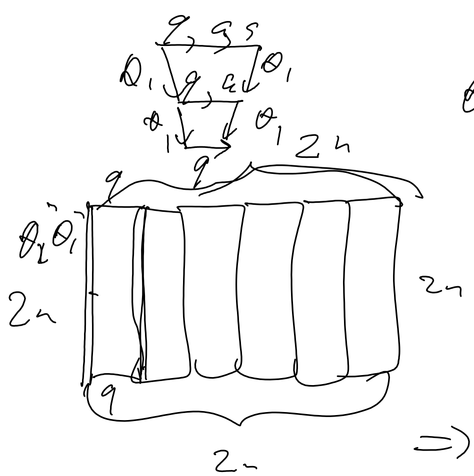
So the total number of cells is at most cubic

$\Rightarrow$ D.f. $\leq n^3/8 + n^2/4$

At least cubic?



Area is quadratic



perimeter is $8n$

Area is $2n^3$

$\Rightarrow$ Dehn function is at least cubic

Theorem The a.s. cones of this group are not simply connected not simply connected

Curt Kent : fund. groups of a.c are uncountable.

Miller machine

Start with any f.p. group $\langle X \mid R\rangle = G$

$\langle X \mid r_1, r_2, \dots, r_n\rangle$

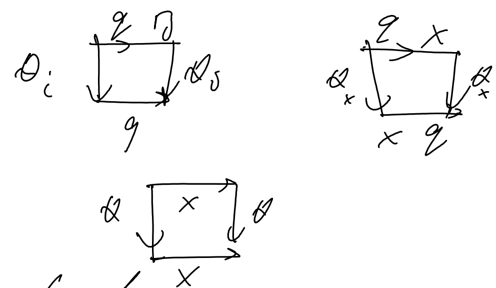
Construct a new group.

$\langle X, q, \theta_1, \theta_2, \dots, \theta_n, \theta_x, x \in X \mid$

$q^{\theta_i} = q r_i$

$q^{\theta_x} = x^{-1} q x$ $\qquad$ $x q^{\theta_x} = q x, \ x \in X$ $\quad$ $(xq)^{\theta_x} = qx$

$x\theta = \theta x \ \rangle$



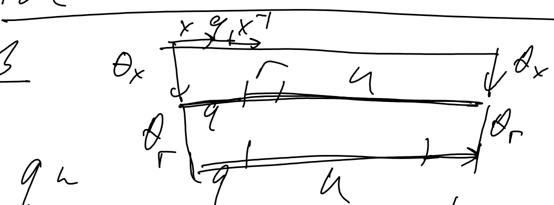
The group is free-by-free

$\langle X, q\rangle$ - free

$\langle \theta_{..}\rangle$ - free

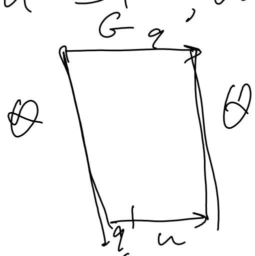
Theorem The Dehn function is cubic

Start with



By conjugating with $\theta_x^{\pm 1}, \theta_r^{\pm 1}$ we can move q anywhere, and insert r

If $u = 1$, then in the Miller group



Theorem (Miller) $qu \sim q$ in Miller group

Theorem (Miller) $qu \sim q$ in Miller group $\iff u =_G 1$.

So if G has "bad" Dehn function then M_G has "bad" conjugacy problem, cubic Dehn function and free-by-free

These are 2 examples of S-machines

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11/9/2012 | S-machines

Miller machine

$G = \langle X \mid R \rangle$

free generators of the HNN

$$M(G) = \langle \underbrace{X, q, \alpha, \omega}_{\text{free group}}, \theta_x, x \in X, \theta_r, r \in R \mid$$

$\alpha^\theta = \alpha, \omega^\theta = \theta$

$q^{\theta_x} \equiv \theta_x^{-1} q \theta_x = x^{-1} q x$ q moves to the left through x

$q^{\theta_r} \equiv \theta_r^{-1} q \theta_r = r q$

$x^\theta = x$ (x commutes with all θ)

$x y q z t$

command letters

$(xy\,qzt) \to (xy\,qzt)^{\theta_y} = x\,qy\,zt \to$

$\to (xqyzt)^{\theta_r} = (xrqyzt)$

$\alpha u q \omega$, $u \in (X \cup X^{-1})^*$ — state

$\alpha u q v \omega$ – configuration

markers.

input configuration

Suppose we have a word in θ's, $W(\theta)$

$(\alpha u q \omega)^{W(\theta)} = \alpha q \omega \implies u = 1$ in G.

Notice: if

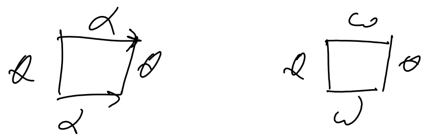
$(\alpha u q v w)^{\alpha} = (\alpha u' q v' w)$, then
$uv = u'v'$ in G

<u>Lemma</u> $u = 1$ in $G \iff \exists\ \underline{w(\theta)} : (\alpha u q w)^{w(\theta)} =$
$= \alpha q w$

<u>Theorem</u> $(\alpha u q w)$ and $(\alpha q w)$ are conjugate in $M(G)$ iff $u = 1$ in G.

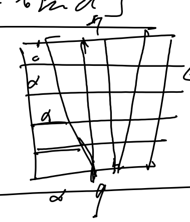
□ Suppose $(\alpha u q w)^{w} = \alpha q w$

$\alpha^{\theta} = \alpha$

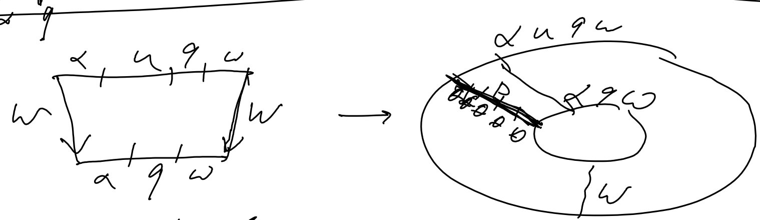


α-bands, ω-bands (∃ α-band starting at the bottom α)

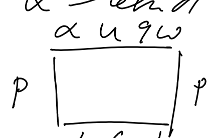
<u>θ-bands</u> do not form cycles



← θ-bands   q-band



Consider α-band



p is a product of θ's

$p = p(\theta)$

$\Rightarrow (\alpha u q w)^{p(\theta)} = \alpha q w \Rightarrow u = 1$ in G.

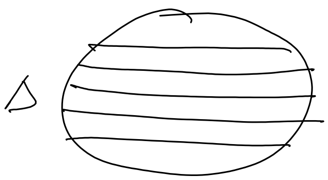
In particular

<u>Cor.</u> If G has undecidable word problem, then $M(G)$ has undecidable conjugacy problem.

$M(G)$ is free-by-free

Th. Dehn function of $M(G)$ is at most cubic

$\Box$ $\quad$ $\#$ θ-bands $\leq \frac{|\partial(\Delta)|}{2}$



$\# q$-bands is $\leq \frac{|\partial(\Delta)|}{2}$

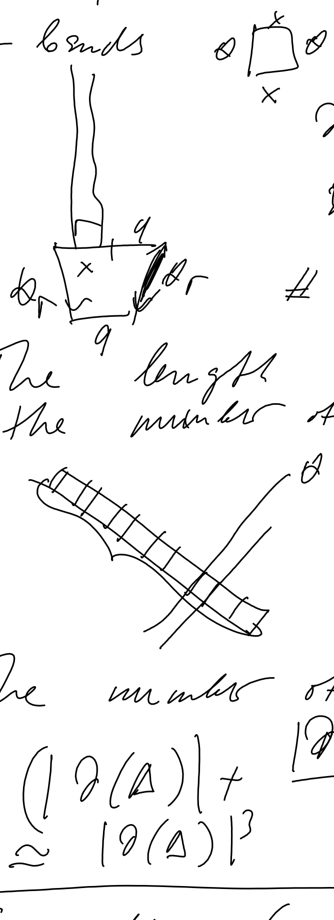
$\#$ α-bands

$\#$ ω-bands

$\Rightarrow$ $\#$ of cells containing q, α, ω is at most

$$\frac{|\partial(\Delta)|^2}{4}$$

x-bands – start and end either on $\partial(\Delta)$ or $\partial(\pi)$, π is a θ-cell.



$\#$ x-bands $\leq |\partial(\Delta)| + \frac{|\partial(\Delta)|^2}{4}$

The length of every x-band is at most the number of θ-bands, $|\partial\Delta|/2$

The number of all cells

$$\left(|\partial(\Delta)| + \frac{|\partial(\Delta)|^2}{4}\right) + \left(|\partial(\Delta)| + \frac{|\partial(\Delta)|^2}{4}\right)\frac{|\partial(\Delta)|}{2}$$

$$\simeq |\partial(\Delta)|^3$$

$\blacksquare$

S-machines (general)

Let us try to simulate a T.M.

Def Hardware, software

X – tape letters.

Q – state letters, $Q \ni \alpha, \omega$.

Q - state letters, $Q \ni \alpha, \omega$.

Configurations $\alpha \underset{\text{tape}}{\sqcup} q_1 \underset{\text{tape}}{\sqcup} q_2 \sqcup \dots \omega$

Software: letters $\theta_1, \dots, \theta_n$ — commands

θ_i locally changes configuration

θ_i: partial maps from set conf. to itself changing conf. locally around state letters.

Set of stop conf usually ones containing certain state letter q_0

This def is too general and too less

More specifically

We define n-tape Turing machine.

A tape is a conf. $\alpha_1 u_1 q_1 v_1 \omega_1$

A conf of the machine is

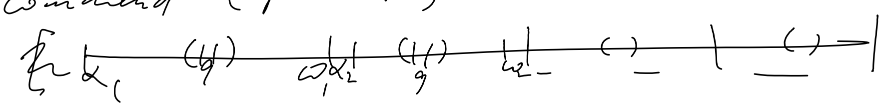
$$(\alpha_1 u_1 q_1 v_1 \omega_1)(\alpha_2 u_2 q_2 v_2 \omega_2)\dots(\alpha_n u_n q_n v_n \omega_n)$$

$$X = X_1 \dot\cup X_2 \dot\cup \dots \dot\cup X_{2n}$$

$$\alpha_i \neq \alpha_j, \quad \omega_i \neq \omega_j$$

$$Q = Q_1 \dot\cup Q_2 \dot\cup \dots \dot\cup Q_n$$

A command (picture)



$$[p_1 q_1 r_1 \to p_1' q_1' r_1', \; p_2 q_2 r_2 \to p_2' q_2' r_2', \dots]$$

Every command is a partial transf. the set of conf (p, r - words in appropriate pieces of X) | $q_i \in Q_i$

There is a natural equivalence of machines

Convenient to assume p_i, r_i consist of at most one letter.

[illegible] Q_i has $q_i^{(0)}$

Every Q_i has $q_i^{(0)}$

The stop conf. $\underline{\alpha_1 q_1^0 \omega_1 \alpha_2 q_2^0 \omega_2 \cdots \alpha_n q_n^0 \omega_n} = \text{Stop}$

A machine recognizes a conf. if there exists a sequence of commands taking it to the stop conf.

1) Time function

2) Space function

$C \xrightarrow{---} \text{stop}$

Time(c) – shortest distance from c to stop

$$\text{Time}(n) = \max_{\substack{|c| \le n \\ c \to \text{stop}}} (\text{Time}(c))$$

Role of the Dehn function

Space(c) – min for all computations of the max length of a config.

$$\text{Space}(n) - \text{space function} = \max_{\substack{|c| \le n \\ c \to \text{stop}}} (\text{Space}(c)) \leftarrow \text{Role of isodism. function}$$

Dehn functions are time functions of some T.M. (Miller machine)

The time to compute $T(n)$ is at most $T(n)$

---

n^α $\forall \alpha > 2$ is ~ the time function of a tm

$n \log n$, $n^{\pi}(\log n)^e$ – time function

Theorem If 1) $T(n) \ge n^4$

2) $T(m+n) \ge T(m) + T(n)$

3) $\sqrt[4]{T(n)}$ must be the time function of a TM.

Then there exists a f.p group with Dehn function $\simeq T(n)$

The set of possible Dehn functions is very large very large

n^α, $\alpha \geq 4$, if α is rat. fast

computable number

The first k digits of α can be computed by a D.T.M. in time at most 2^{2^k}

$n^{\pi+1}$ — Dehn functions

$n^{\pi+1}(\log n)^{e}$ —

Proof M has time function $\sqrt[4]{T(n)}$

$M \longrightarrow$ group.

Step 1 $G_1 = \langle X \cup Q \cup \{\vec{\theta}, \dots\} \rangle$

Relations

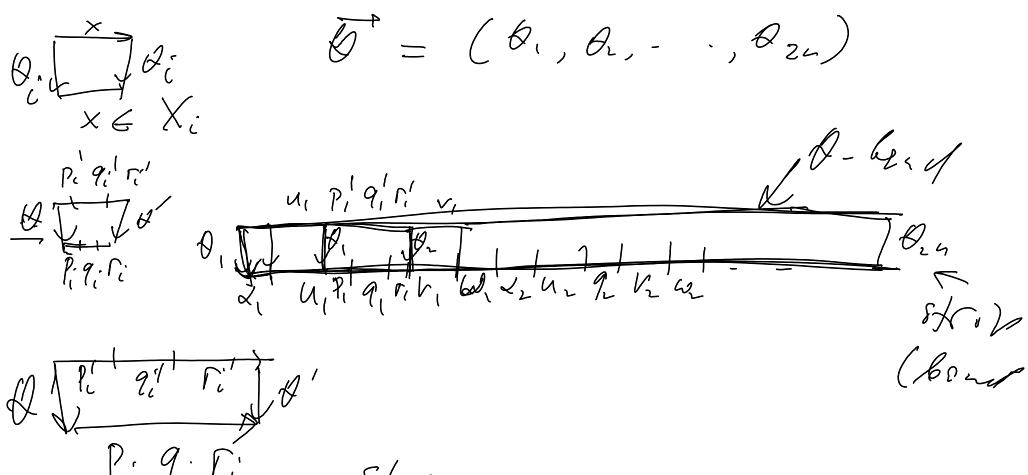
$x^\theta = x$, $(p_i q_i r_i)^\theta = (p_i' q_i' r_i')$

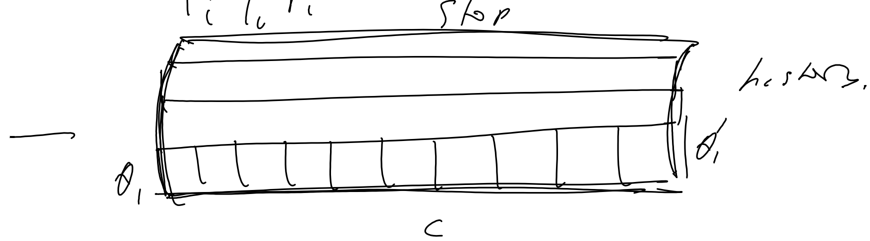
$\alpha_1\ u_1\ q_1\ v_1\ \omega_1\ \alpha_2\ u_2\ q_2$

$\theta = [p_1 q_1 r_1 \to p_1' q_1' r_1', \dots]$

$\vec{\theta} = (\theta_1, \theta_2, \dots, \theta_{2n})$

$x \in X_i$

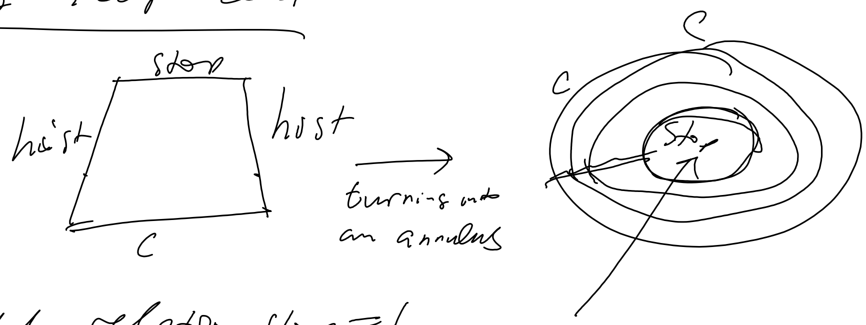




$\theta = \theta'$ (equalize them)

For others accept conf

For every accept conf



Then add relation stop $=1$

The new group is G_2

Lemma If C is accepted, then $C=1$ in G. (The area is at most Time·Space)