Lecture 4.

(X_i, d_i), $o_i \in X_i$,

ω - ultrafilter.

$$\Pi_b (X_i, d_i) = \{ (x_i), \ \underline{d}((x_i),(o_i)) < \infty \}$$

$$d((x_i),(y_i)) = \lim^\omega (d_i(x_i, y_i))$$

$$\lim^\omega (X_i, d_i) = \Pi_b (X_i, d_i) / \sim$$

$$(x_i) \sim (y_i) \iff d = 0$$

$$X_i = X/p_i = (X, d/p_i)$$

$$\lim^\omega (X_i, d/p_i) = \mathrm{Con}^\omega (X, (p_i), (o_i))$$

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1) X - homogeneous $\Rightarrow$ Con^ω $\underline{\text{is}}$ ((o_i) - independent)

$G \curvearrowright X$, $\Pi_b G/\omega \curvearrowright \mathrm{Con}^\omega$

2) $X \underset{q.i.}{\sim} Y$

$$\varphi : X \to Y$$

$$\frac{1}{\lambda} \frac{d}{p_i} - \frac{C}{p_i} \leq \frac{d(\varphi(x), \varphi(y))}{p_i} \leq \lambda \frac{d(x,y)}{p_i} + \frac{C}{p_i}$$

$$\mathrm{Con}^\omega (X) \underset{\lambda\text{-b.L.}}{\sim} \mathrm{Con}^\omega (Y)$$

3) $\mathrm{Con}^\omega (X)$ - complete

4) $Y_i \subseteq X$; $\lim^\omega (Y_i)$

$\gamma_i = q_i[0, \ell_i]$

$\Rightarrow$ $\lim \gamma_i$ is a path $q_i : [0, \lim^\omega \ell_i / p_i]$

$(q_i(t_i)) \overset{def}{=} q(\lim^\omega t_i)$

limit of geod is geod.

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5) $Con^\omega(X) \supseteq \bigcirc \sim \bigcirc = \lim \bigcirc_i \subseteq X$

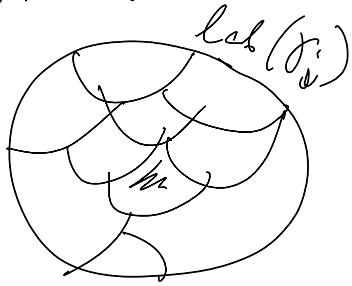
Th (Gromov) $G = \langle X \rangle$. If all a.c. of G are simply connected, then

a) G is f.p.

b) G has polynomial Dehn function

Pf Suppose G is not f.p. $\Rightarrow$

$\exists\ \gamma_i$, $lab(\gamma_i)$ is not a corollary of shorter words (relations)



$p_i = |\gamma_i|$

$Con^\omega(G, (p_i)) = \lim^\omega (X/p_i)$

γ_i has length 1 in X/p_i)

$\gamma = \lim^\omega \gamma_i$, γ is a loop in Cone of length 1.

$\exists$ cont. map $\varphi : D^2 \to$ Cone

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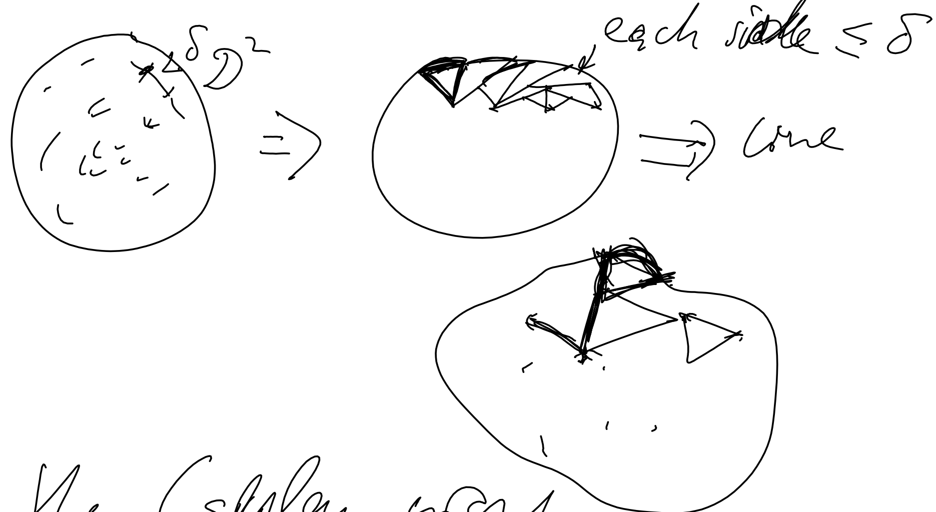
$S^1 \to \gamma$

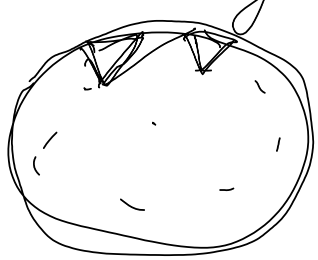
D^2 is compact

φ is uniformly continuous

Take $\varepsilon = \frac{1}{5}$, find δ



In the Cayley graph



γ_i perimeter of each triangle is $\frac{3}{5}|\gamma_i| + o(|\gamma_i|)$
$\leq \frac{4}{5}|\gamma_i|$

Lab(γ_i) is a relation which follows from relations of length $\leq \frac{4}{5}|\gamma_i|$

ω - a. s. | $O^{\omega}(p_i)$

Poly Dehn function

$f(x)$ is poly. $\Rightarrow$

$f(2x) < K f(x)$

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Suppose there is no polynomial s.t.
$\mathrm{Dehn}(n) \leq f(n)$

$\Rightarrow \forall K \ \exists \gamma_K$ $\boxed{\mathrm{Area}(\gamma_K) \geq K\, \mathrm{Area}(\mu)}$

$|\mu| < \dfrac{|\gamma_K|}{2}$

Do the same as above, subdivide
D^2 into 10^{70} of little triangles $\Rightarrow$
$K = 10^{70}$ is a contradiction

Hyperbolic groups

Def  ← geod — is in δ-neigh. of — $\Rightarrow$ δ-hyperbolic

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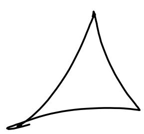
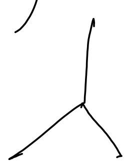
Divide the dist by 2 $\Rightarrow$

 $\Rightarrow$ is in $\delta/2$-neigh. of

$\lim^{\omega}(X/p_i)$ — complete 0-hyp. geod space



The a.c. of every hyp. space is an $\mathbb{R}$-tree

If X is not q.i. to $\mathbb{Z}$

then a.c. ~~are~~ all isometric to the

universal $\mathbb{R}$-tree – $\mathbb{R}$-tree of degree continuum at every vertex

(Erschler + Polterovich).

Th A group is hyperbolic iff all a.c. are trees.

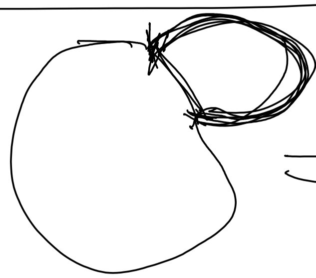
Corollaries

1) Approximation (Delzant)
2) Prove all unproved statements in Gromov's paper "Hyp. groups"

Th Every hyp. group has Dehn presentation and a linear Dehn function

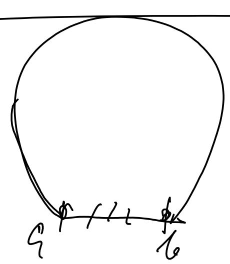
A presentation $G=\langle X \mid R\rangle$ is Dehn if. Every word $W=1$ in G has more than a half of a relator

Dehn.



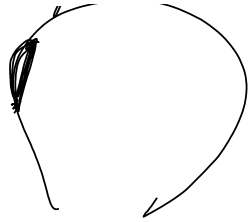
$\Rightarrow$ linear isop. inequality (Dehn function)

Bump



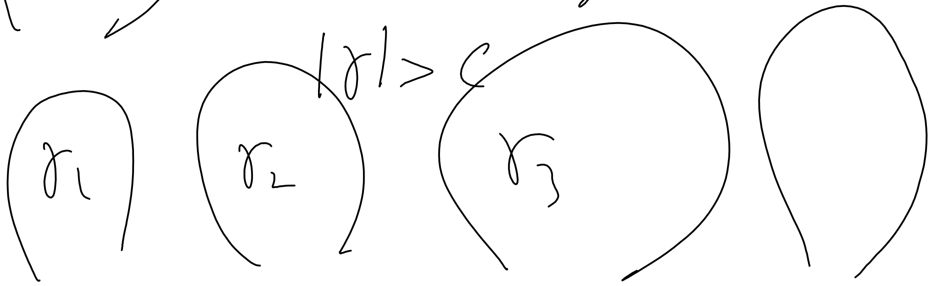
$|a-b| \leq \frac{1}{3}|\gamma|$

Theorem Every bump γ has a subbump of length $\geq \frac{|\gamma|}{3} > 0$



(true for all suff large γ $|\gamma| > C$

D^{δ}



length $(\gamma_i) = p_i$

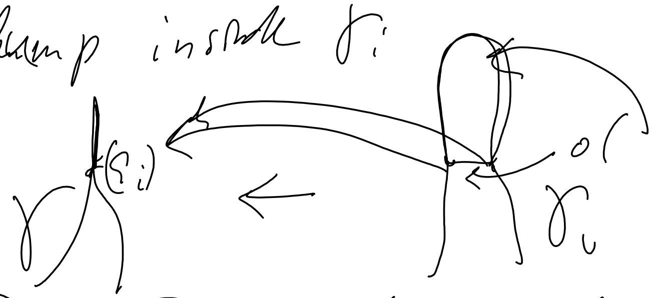
$X/p_i \Longrightarrow Con(X/p_i)$

$\lim^{\omega} \gamma_i$ is a bump γ in Cone

γ is a bump at a tree T

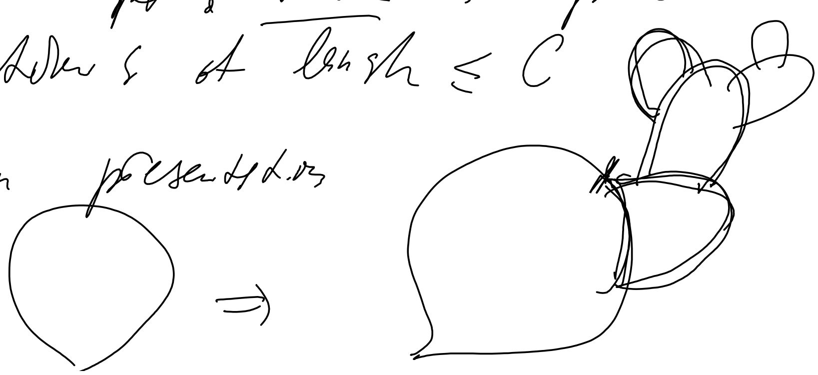
$[(a_i),(b_i)] = \lim \delta_i$, $\delta_i \subseteq \gamma_i$

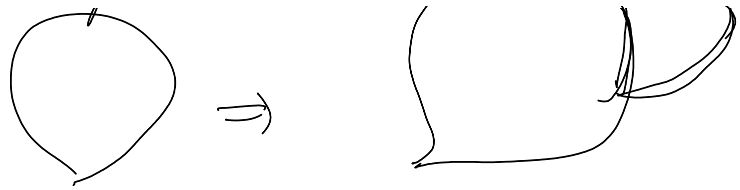
then the subpath (a_i, b_i) is a bump inside γ_i



$\Longrightarrow$ Every loop of length $\geq C$ has a bump. Take as a presentation all relations of length $\leq C$

$\Rightarrow$ Dehn presentations





G is hyp. $\Longleftrightarrow$ Dehn function is linear

Th. There is a universal constant C such that if the Dehn function is $D(n) \leq Cn^2 \Rightarrow$ group is hyperbolic.

No Dehn functions between n^1 and n^2

Lecture 5. Quadratic Dehn functions