Lecture 4. (X_i, d_i), $o_i \in X_i$,

ω - ultrafilter.

$$\Pi_b(X_i, d_i) = \{(x_i),\ \underline{d}((x_i),(o_i)) < \infty\}$$

$$d((x_i),(y_i)) = \lim{}^\omega (d_i(x_i, y_i))$$

$$\lim{}^\omega (X_i, d_i) = \Pi_b(X_i, d_i)/\sim$$

$$(x_i) \sim (y_i) \iff d = 0$$

$$X_i = X/p_i = (X, d/p_i)$$

$$\lim{}^\omega (X_i, d/p_i) = \mathrm{Con}^\omega(X, (p_i), (o_i))$$

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1) X – homogeneous $\Rightarrow$ Con^ω is (o_i)– independent

$G \curvearrowright X$, $\Pi_b G/\omega \curvearrowright \mathrm{Con}^\omega$

2) $X \underset{q.i.}{\sim} Y$ $\quad \varphi : X \to Y$

$$\frac{1}{\lambda}\frac{d}{p_i} - \frac{C}{p_i} \le \frac{d(\varphi(x), \varphi(y))}{p_i} \le \lambda \frac{d(x,y)}{p_i} + \frac{C}{p_i}$$

$$\mathrm{Con}^\omega(X) \underset{\lambda\text{-b.L.}}{\sim} \mathrm{Con}^\omega(Y)$$

3) $\mathrm{Con}^\omega(X)$ – complete

4) $Y_i \subseteq X$; $\lim{}^\omega (Y_i)$

$Y_i = q_i[0, \ell_i]$

$\Rightarrow$ $\underline{\lim Y_i}$ is a path $q : [0, \lim{}^\omega \ell_i/p_i]$

$(q_i(t_i)) \overset{\text{def}}{=} q(\lim{}^\omega t_i)$

limit of geod is geod.

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5) $\mathrm{Con}^\omega(X) \supseteq \bigcirc \sim \bigcirc = \lim \bigcirc_i \subseteq X$

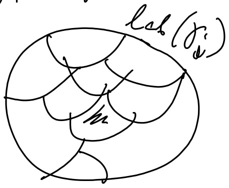
Th (Gromov) $G = \langle X \rangle$. If all a.c. of G are simply connected, then

5) G is f.p.

6) G has polynomial Dehn function

Pf Suppose G is not f.p $\Rightarrow$

$\exists\ \gamma_i$ $\quad$ $\mathrm{Cab}(\gamma_i)$ is not a corollary of shorter words (relations)



$p_i = |\gamma_i|$

$\mathrm{Con}^\omega(G, (p_i)) = \lim^\omega (X/p_i)$

γ_i has length 1 in X/p_i)

$\gamma = \lim^\omega \gamma_i$, γ is a loop in Cone of length 1.

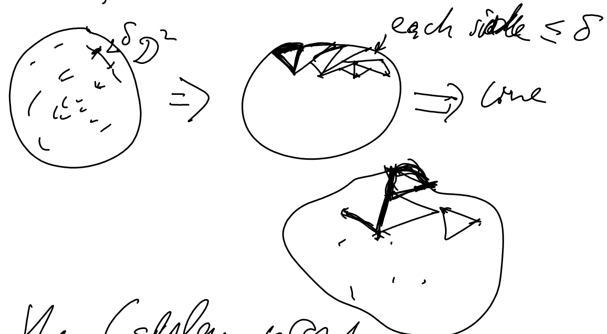
$\exists$ cont. map $\varphi: D^2 \to$ Cone

$S^1 \to \gamma$

D^2 is compact

φ is uniformly continuous

Take $\varepsilon = \frac{1}{5}$, find δ



In the Cayley graph

perimeter of each triangle is $\frac{3}{5}|\gamma_i| + o(|\gamma_i|)$

$\le \frac{4}{5}|\gamma_i|$

lab(γ_i) is a relation which follows from relations of length $\le \frac{4}{5}|\gamma_i|$

ω-a.s. | $O^\omega(p_i)$

Poly Dehn function

$f(x)$ is poly. $\Rightarrow$

$f(2x) < K f(x)$

Suppose there is no polynomial s.t.

$\mathrm{Dehn}(n) \le f(n)$

$\Rightarrow \forall K \ \exists \gamma_K$ $\boxed{\mathrm{Area}(\gamma_K) \ge K\, \mathrm{Area}(\gamma)}$

$|\gamma|$

$$|\mu| < \frac{|\gamma_K|}{2}$$

Do the same as above, subdivide D^2 into 10^{70} of little triangles $\Rightarrow$
$K = 10^{70}$ is a contradiction

Hyperbolic groups

Def △ geod) is in δ-neigh. of ⟩ $\Rightarrow$ δ-hyperbolic

Divide the dist by 2 $\Rightarrow$

△ $\Rightarrow$) is in $\delta/2$-neigh. of ⟩

$\lim^\omega (X/p_i)$ – complete 0-hyp. geod space

The a.c. of every hyp. space is an $\mathbb{R}$-tree

If X is not q.i. to $\mathbb{Z}$
then a.c. are all isometric to the universal $\mathbb{R}$-tree – $\mathbb{R}$-tree of degree continuum at every vertex

(Erschler + Polterovich).

Th A group is hyperbolic iff all a.c. are trees.

Corollaries
1) Approximation (Delzant)
2) Prove all unproved statements in Gromov's paper "Hyp. groups"

Th Every hyp. group has Dehn presentation and a linear Dehn function

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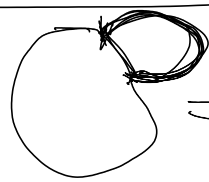
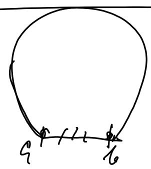
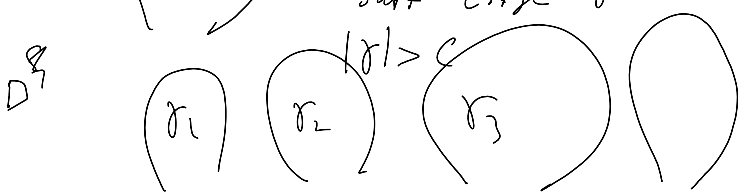
A presentation $G = \langle X \mid R \rangle$ is Dehn if: Every word $W = 1$ in G has more than a halfpath of a relator

Dehn.  $\Rightarrow$ linear isop. inequality (Dehn function)

Bump  $|a-b| \leq \frac{1}{3}|\gamma|$

Theorem Every bump γ has a subbump of length $\geq \frac{|\gamma|}{3} > 0$

(true for all suff large γ $|\gamma| > C$

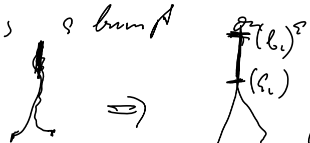


length $(\gamma_i) = p_i$

$X/p_i \Longrightarrow Con(X/p_i)$

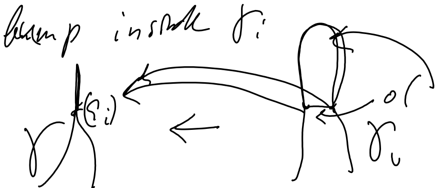
$\lim^{\omega} \gamma_i$ is a bump γ in Cone

γ is a bump a tree T



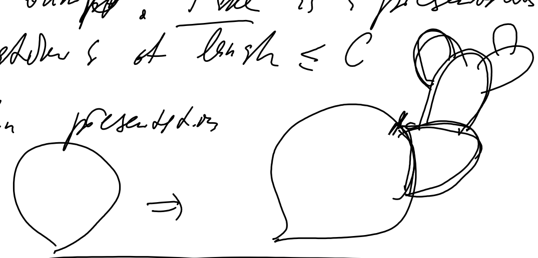
(b_i) (a_i) $[(a_i),(b_i)] = \lim \delta_i$, $\delta_i \subseteq \gamma_i$

then the subpath (a_i, b_i) is a bump inside γ_i



$\Rightarrow$ Every loop of length $\geq C$ has a bump. Take as a presentation all relations of length $\leq C$

$\Rightarrow$ Dehn presentation



G is hyp. $\Leftrightarrow$ Dehn function is linear

There is a universal constant C such that if the Dehn function is $\varnothing(n) \le Cn^2 \Rightarrow$ group is hyperbolic.

No Dehn functions between n^1 and n^2

Lecture 5. Nilpotent groups. Quadratic Dehn functions

Th (Papasoglu) If $A\delta$-function (of all a.c. of a group) is bounded by n^c, then the Dehn function of the group $\le n^{c+\varepsilon}$ for $\forall \varepsilon$. every ε. (That is for every ε there exists a constant C_ε so that the Dehn function $\le C_\varepsilon n^{c+\varepsilon}$)

Drutu used this theorem to estimate Dehn functions of nilpotent groups

Def: N - nilpotent group f.g. $[x, y, z, \ldots] = 1$, here $[x,y,z] = [[x,y],z]$, $[x,y] = x^{-1}y^{-1}xy$

Th. For every nilpotent group N every a.c. $Con^\omega(N,(p_i))$ is a Lie group constructed as follows

$N/tor(N)$ - torsion - free -

$Tor(N)$ - all elements of finite order

Co-compact lattice of $\bar{N}$

To construct $\bar{N}$ add formal powers x^α, $\alpha \in \mathbb{R}$ for all elements x of N

The a.c. is the homogeneous Lie group constructed from $\bar{N}$

$Hom(\bar{N})$

$\bar{N} > [\bar{N}, \bar{N}] > \ldots \quad 1$

$\bar{N} > c_1(\bar{N}) > c_2(\bar{N}) > \ldots 1$

$\bar{N}/c_1(\bar{N}) \oplus \ldots \oplus c_k(\bar{N})/c_{k+1}(\bar{N}) \leftarrow$ This is the set $Hom(\bar{N})$

Operation $\rightarrow [g\, c_i(N), h\, c_j(N)] =$

$= [g, h]\, C_{i+j}(N)$

$N > c_1(N) > c_2(N) > \ldots$

$X_1 \qquad X_2$

$[x, y] =$

$x \in C_i/C_{i+1} \qquad y \in C_j/C_{j+1}$

$[x,y] \in C_{i+j}$

$Hom(\bar{N})$ is homogeneous because for every $x \in C_i(H) \setminus C_{i+1}(H)$, $y \in C_j(H) \setminus C_{j+1}(H)$, $[x,y] \in C_{i+j} \setminus C_{i+j+1}$ or $[x,y] = 1$

Pittet: $A\delta$ for $Hom(N)$ is $\le n^{k+1}$ where k is the class of nilpotency

$N > [N, N] > 1$

The metric is Carnot-Carathéodory

http://intlpress.com/JDG/archive/1985/21-1-35.pdf

horizontal $\nearrow x_1, x_2, \ldots, x_k \quad y_1, \ldots, y_s \leftarrow$ vertical

$c = [a, b]$, $c^{n^2} = [a^n, b^n]$; to travel n^2 up one can instead travel $4n$ horizontally

$\Rightarrow$ Th (Drutu) Every nilp. class c group has Dehn function at most $n^{c+1+\varepsilon}$ (use Papasoglu)

Gersten + Rielly: n^{c+1} (Also Gromov)

[illegible]

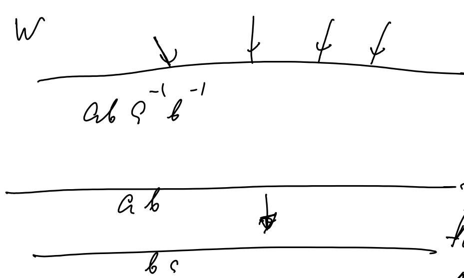
Free nilp. group of class c has Dehn function n^{c+1}

$H_3 = \langle a, b, c \mid [a,b]=c,\ ca=ac,\ cb=bc \rangle$ ← Cubic Dehn function

$H_5 = \langle a, b, x, y, c \mid [a,b]=c=[x,y]=c,\ [c,a]=[c,b]=[c,x]=[c,y]=1,\ \{a,b\} \leftrightarrow \{x,y\} \text{ commute} \rangle$

Dehn function is quadratic (Allcock, Olshanskii–Sapir)

Baby example $\mathbb{Z}^2 = \langle a, b \mid [a,b]=1 \rangle$



W=1 if and only if by inserting relation and reducing we get an empty word. We can at every step replace ab by ba. The number of steps – Dehn function. How many steps needed – at most $|W|^2$: move first letter of w to the right until it meets a^{-1}, etc. So the Dehn function of $\mathbb{Z}^2$ is quadratic

$H_5/\langle c \rangle = \langle a, b, x, y \rangle$ free Abelian

$W =_{H_5} 1 \Rightarrow W$ is a commutator word.

$\Rightarrow W = c^s$, $W =_{H_5} [a,b]^p [x,y]^q$

Use commutativity in H_5, $\{a,b\} \sim \{x,y\}$:

$W \approx W'_{ab} W''_{xy} = W'_{ab} W''_{ab} =$

Th The Dehn function of H_5 is quadratic

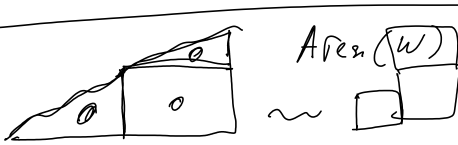
Lemma One can transform W_{ab} into W_{xy} in quadratic number of steps

It is enough to consider words in a, b

$W_W \quad W W_b^{-1} W_a^{-1} = c^s$

Define the area of any word $W(a,b)$

Area(W) ~



Divide the triangle into 3 parts

Replace each part by a rectangle, replace a product of two rectangles by one rectangle

Ex $C_{10} = \langle a_1, b_1, \dots, a_5, b_5 \mid [a_1,b_1][a_2,b_2]\dots[a_5,b_5]=1 \rangle$

[illegible]

of two rectangles by one rectangle.

$C_{10} \times C_{10} / \{(a, u^{\pm}_{2}u)\} = G_{10}$

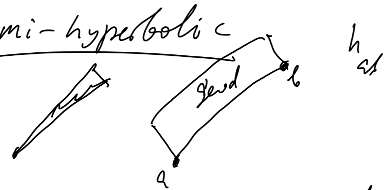
G_{10} has Dehn function at most $n^2 \log n$

Wenger: Dehn function of G_{10} is $\not\approx n^2$

Quadratic Dehn functions

Hyp. groups linear Dehn function

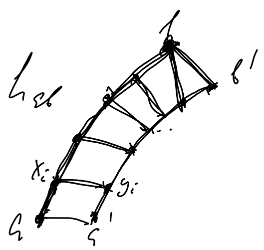
Semi-hyperbolic



$\forall a, b \;\exists$ geodesic h_{ab}

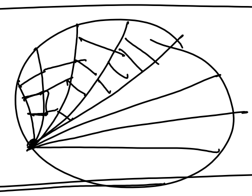
(i) $h_{ga, gb} = g\, h_{ab}$

(ii)



$$\operatorname{dist}(x_i, y_i) < C\left(\max\{\operatorname{dist}(a, a'), \operatorname{dist}(b, b')\}\right) + C_1$$

Statement: Every semihyp. group has quadratic Dehn function



Th The following groups have quadr. Dehn functions

1) Drutu Polycyclic groups.

$\mathbb{Q} \subseteq K$ - totally real.

$\mathcal{O}(K) = \mathbb{Z}^n$ | $\mathbb{Z}^n \rtimes \mathbb{Z}^d$

$U(K) = \mathbb{Z}^d$

has Dehn function n^2

2) Cornulier - Tessera $\exists$ nonabelian group with quadr Dehn function $\supseteq BS(1,2)$.

3) $F_k \rtimes \mathbb{Z}$ (Bridson + Groves)

- quadr Dehn function

4)

$F_k \rtimes \mathbb{Z}$ $\quad \langle x_1, \ldots, x_k, t$

$\underline{t^{-1} x_i t} = u_i$