A Combination Theorem for Affine Tree-Free groups

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12 May 2016

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Equivalently, a Λ-tree is a geodesic, 0-hyperbolic Λ-metric space such that $(x \cdot y) \in \Lambda$.

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A finitely generated freely indecomposable $\mathrm{ITF}(\mathbb{R})$ group is free abelian or a residually free surface group. (Rips Theorem)

Other examples of ITF groups:

1. locally fully residually free groups
2. certain 'length-preserving' HNN extensions of other ITF groups. For example, $\langle x, y, z | x^2y^2z^2 = 1\rangle \cong \langle u, z, t | tut^{-1} = z^{-2}u^{-1}\rangle$ is $\mathrm{ITF}(\mathbb{Z} \times \mathbb{Z})$.

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However

$$\langle x, y, t | t[x, y]t^{-1} = [x^2, y^2]\rangle$$

is not ITF: $\ell[x, y]$ is always less than $\ell[x^2, y^2]$ if x and y are hyperbolic and $xy \neq yx$.

Finitely presented ITF groups

- are $\mathrm{ITF}(\mathbb{R}^n)$ for some n
- are right orderable (Chiswell)
- are biautomatic
- are hyperbolic relative to non-cyclic abelian subgroups (using Dahmani's Combination Theorem)
- admit a quasi-convex hierarchy
- are virtually special (using Wise's results)
- are virtually orderable (but not necessarily orderable - OR)
- are linear, and hence residually finite
- have solvable Word, Conjugacy and Isomorphism Problems.

See Kharlampovich, Myasnikov, Serbin Actions, length functions, and non-archimedean words IJAC, 2013.

Theorem (Bass 1991)

An isometric action on a Λ-tree gives rise to

- *a graph of groups decomposition;*
- *isometric actions of the vertex groups on Λ_0-trees.*

These satisfy compatibility conditions

1. *edge stabilisers $\mathcal{G}(e)$ match up with end stabilisers $(\mathcal{G}(x^*))_{\epsilon_e}$ where $x^* = \partial_0 e$. Also ends ϵ_e are of full Λ_0-type.*
2. *if $x^* = \partial_0 e = \partial_0 f$ with $e \neq f$ then ϵ_e and ϵ_f lie in distinct $\mathcal{G}(x^*)$-orbits.*
3. $\tau_e\ \alpha_e(s) + \tau_{\bar{e}}\ \alpha_{\bar{e}}(s) = 0$ *for $s \in \mathcal{G}(e)$ [compatibility of directions and translation lengths of elements of $\mathcal{G}(e)$.]*

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And conversely.

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Liousse (2001): Examples of groups that admit free affine actions on $\mathbb{R}$-trees, but that have no free isometric action on any $\mathbb{R}$-tree. E.g.

$$\langle x_1, x_2, x_3, y_1, y_2, y_3 \mid [x_1, y_1] = [x_2, y_2] = [x_3, y_3] \rangle$$

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Note that if $\Lambda = \mathbb{Z} \times \Lambda_0$ then β_g is determined by its effect on $(1, \lambda_0)$ $(\lambda_0 \in \Lambda_0)$, and $\beta_g(1, \lambda_0) = (1, \theta_g \lambda_0 + \mu_g)$ for some $\theta_g \in \mathrm{Aut}^+(\Lambda_0)$

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4. more generally the groups $T^*(n, \mathbb{R})$ of upper triangular matrices with positive diagonal entries.

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Isometric case: to extend isometric actions of Γ_{x^*} on Λ_{x^*}-trees $X(x^*)$ to an isometric action of $\Gamma = \pi_1(\mathcal{G}, Y^*)$, we need:

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2. the ends ϵ_e of the $0 \times \Lambda_0$-balls to be of full Λ_0-type
3. matching hyperbolic lengths of edge group elements that is, $\ell(\alpha_e(g)) = \ell(\alpha_{\bar{e}}(g))$ for $g \in \Gamma_e$.

Theorem (OR)

Bass's Theorem goes through with 'isometric' replaced by 'affine' if we modify the compatibility conditions appropriately.

Key compatibility condition in affine case:

$$\theta_{g_e}\left[\theta_{\alpha_e(s)}\delta_e^{Y^*}(x) - \delta_e^{Y^*}\cdot\alpha_e(s)(x)\right] + \theta_{g_{\bar{e}}}\left[\theta_{\alpha_{\bar{e}}(s)}\delta_{\bar{e}}^{Y^*}(y) - \delta_{\bar{e}}^{Y^*}\cdot\alpha_{\bar{e}}(s)(y)\right] = \mu_{\alpha_{|e|}(s)}$$

So suppose a graph Y^* is given, together with ATF groups $\mathcal{G}(x^*)$ for each vertex $x^* \in Y^*$. Assume that edge groups act isometrically, and are cyclic. What obstacles are there to producing a free affine action on some Λ-tree?

1. **Need a common Λ_0 for all x^*.**
   Solution: Embed all given Λ_{x^*} in, say, $\Lambda_0 = \oplus_{x^* \in Y^*} \Lambda_{x^*}$. We can then replace the given homomorphisms into $\mathrm{Aut}^+(\Lambda_{x^*})$ by homomorphisms into $\mathrm{Aut}^+(\Lambda_0)$ in an obvious way.

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3. **If e is the edge joining x^* to y^*, we need the ends ϵ_e and $\epsilon_{\bar{e}}$ joining the balls $X(x^*)$ and $X(y^*)$ of radius $(0\times)\Lambda_0$ to be of full $0 \times \Lambda_0$-type.**
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   Snag: The natural extension of a free action may not be free.
   Solution: Restrict attention to essentially free actions. . .

Each non-trivial g has an axis A_g, which is linear, and hence embeds in Λ. $\iota : A_g \to \Lambda$. (ι is not unique.)

Now $\iota(gx) = \beta_g \iota(x) + \nu_g$ for some constant ν_g (independent of x, but not of ι).

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Free isometric actions are automatically essentially free:
$1 - \beta_g = 0$ for all isometries g.

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Solution: If the edge e belongs to the maximal subtree T^*, then we can adjust the given metric $d_{\partial_0 e}$ on $X(\partial_0 e)$: replace $d_{\partial_0 e}$ by $\eta d_{\partial_0 e}$.
If the original action on $(X(\partial_0 e), d_{\partial_0 e})$ was $\theta^{\partial_0 e}$-affine, the action on $(X(\partial_0 e), \eta d_{\partial_0 e})$ will now be $\eta\theta^{\partial_0 e}\eta^{-1}$-affine.

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Further snag: What if there is no o-automorphism of Λ_0 that maps $\ell(u)$ to $\ell(v)$?

Take a regular embedding $h : \Lambda_0 \to \Lambda_1$: that is, an embedding h together with an embedding $\theta_g \mapsto \bar{\theta}_g$ of $\mathrm{Aut}^+(\Lambda_0)$ in $\mathrm{Aut}^+(\Lambda_1)$ such that

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Lemma

1. *The natural extension of an essentially free action to the Λ_0-fulfilment is essentially free.*
2. *Regular embeddings always exist.*
3. *Essential freeness is preserved by regular embeddings.*

Consider $\Gamma = \langle G, t \mid tut^{-1} = v \rangle$ where G is $\mathrm{ITF}(\Lambda_0)$.
In order for Γ to be ITF we would need to have $\ell(u) = \ell(v)$ with respect to some free isometric action of G. But this is impossible in certain cases (as noted earlier).
In order for Γ to be $\mathrm{ATF}(\mathbb{Z} \times \Lambda_0)$ it suffices to find a free isometric action of G on a Λ_0-tree such that

1. u and v generate maximal cyclic subgroups of G
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Taking a regular embedding of Λ_0 in some Λ_1, and replacing Λ_0 by Λ_1, we can ensure that θ_t can be found.

In particular, conjugacy pinched one-relator groups $\langle F, t \mid tut^{-1} = v \rangle$ with maximal cyclic $\langle u \rangle$ and $\langle v \rangle$ and u not conjugate to v^{-1} admit a free affine action on a $\mathbb{Z} \times \mathbb{Q}$-tree.

In particular, conjugacy pinched one-relator groups $\langle F, t \mid tut^{-1} = v\rangle$ with maximal cyclic $\langle u\rangle$ and $\langle v\rangle$ and u not conjugate to v^{-1} admit a free affine action on a $\mathbb{Z} \times \mathbb{Q}$-tree.

For example the groups

$$\langle x, y, t \mid t[x^m, y^n]t^{-1} = [x^r, y^s]\rangle \quad m, n, r, s \neq 0$$

and

$$\langle x, y, t \mid txt^{-1} = [x, y]\ \rangle$$

are $\mathrm{ATF}(\mathbb{Z} \times \mathbb{Q})$.

In particular, conjugacy pinched one-relator groups $\langle F, t \mid tut^{-1} = v \rangle$ with maximal cyclic $\langle u \rangle$ and $\langle v \rangle$ and u not conjugate to v^{-1} admit a free affine action on a $\mathbb{Z} \times \mathbb{Q}$-tree.

For example the groups

$$\langle x, y, t \mid t[x^m, y^n]t^{-1} = [x^r, y^s] \rangle \quad m, n, r, s \neq 0$$

and

$$\langle x, y, t \mid txt^{-1} = [x, y] \rangle$$

are $\mathrm{ATF}(\mathbb{Z} \times \mathbb{Q})$.

(Aside: the latter is even isometrically tree-free, but not residually nilpotent.)

Free affine action of Γ on a Λ-tree X ($\Lambda = \mathbb{Z} \times \Lambda_0$), and where the induced action on the quotient $\mathbb{Z}$-tree X^* is without inversions

$\rightsquigarrow$ graph of groups $\mathcal{G}$: vertex groups $\mathcal{G}(x^*)$ are $\mathrm{ATF}(\Lambda_0)$, edge groups are line stabilisers.

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End stabilisers in $\mathrm{ATF}(\mathbb{Z}^n)$ groups coincide with line stabilisers, which embed in $\mathrm{UT}(n+1, \mathbb{Z})$.

So edge groups are maximal nilpotent.

Theorem

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1. *solvable Word Problem (B. Farb)*
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An $\mathrm{ATF}^o(\mathbb{Z}^n)$ group — that is, a finitely generated group that admits a free affine action on a $\mathbb{Z}^n$-tree where no line has its orientation reversed — is relatively hyperbolic with torsion-free nilpotent parabolic subgroups.
This follows from Bigdely-Wise.

Therefore, finitely generated $\mathrm{ATF}^o(\mathbb{Z}^n)$ groups have solvable Word, Conjugacy and Isomorphism Problems.

It also follows from Dahmani's work that finitely generated $\mathrm{ATF}^o(\mathbb{Z}^n)$ groups are locally relatively quasiconvex.

Thank you!