From local to global conjugacy in relatively hyperbolic groups

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Relative presentations

Let G be a group, $\mathbb{P} = \{P_\lambda\}_{\lambda\in\Lambda}$ a collection of subgroups of G, X a subset of G. We say that X is a *relative generating set of G with respect to* $\mathbb{P}$ if

$$G = \langle (\bigcup_{\lambda\in\Lambda} P_\lambda) \cup X \rangle.$$

In this situation G can be regarded as a quotient group of

$$\overline{F} = \big(\mathop{*}_{\lambda\in\Lambda} \widetilde{P}_\lambda \big) * F(X),$$

where $\widetilde{P}_\lambda$ is a copy of P_λ such that the union of all $\widetilde{P}_\lambda \setminus \{1\}$ and X is disjoint.

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$$\widetilde{P}_\lambda = \langle \widetilde{P}_\lambda \setminus \{1\} \mid \widetilde{\mathcal{S}}_\lambda \rangle,$$

where $\widetilde{\mathcal{S}}_\lambda$ is the set of all words over the alphabet $\widetilde{P}_\lambda \setminus \{1\}$ that represent 1 in the group $\widetilde{P}_\lambda$. Denote

$$\widetilde{\mathcal{P}} = \bigcup_{\lambda\in\Lambda} (\widetilde{P}_\lambda \setminus \{1\}), \quad \widetilde{\mathcal{S}} := \bigcup_{\lambda\in\Lambda} \widetilde{\mathcal{S}}_\lambda.$$

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$$\overline{F} = \langle \widetilde{\mathcal{P}} \sqcup X \mid \widetilde{\mathcal{S}} \rangle$$

and G has a presentation (called relative with respect to $\mathbb{P}$)

$$G = \langle \widetilde{\mathcal{P}} \sqcup X \mid \widetilde{\mathcal{S}} \sqcup \mathcal{R} \rangle.$$

Finite relative presentations

The relative presentation

$$G = \langle \widetilde{\mathcal{P}} \sqcup X \,|\, \widetilde{\mathcal{S}} \sqcup \mathcal{R} \rangle$$

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$$G = \langle X, \mathbb{P} \,|\, \mathcal{R} \rangle.$$

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Example. Consider the amalgamated product

$$G = H_1 *_{K \overset{\alpha}{\to} L} H_2.$$

With $\mathbb{P} = \{H_1, H_2\}$, there is the following relative presentation

$$G = \langle \emptyset, \mathbb{P} \,|\, k = \alpha(k)\, (k \in K) \rangle.$$

It can be chosen finite if K is finitely generated.

Relative isoperimetric functions

Suppose that G has a relative presentation

$$G = \langle X, (P_\lambda)_{\lambda\in\Lambda} \mid \mathcal{R}\rangle. \tag{1}$$

Then G is a quotient of

$$\overline{F} = \left(\mathop{*}_{\lambda\in\Lambda} \widetilde{P}_\lambda \right) * F(X)$$

If a word $W \in (X \cup \widetilde{\mathcal{P}})^*$ represents 1 in G, there exists an expression

$$W \stackrel{\overline{F}}{=} \prod_{i=1}^{k} f_i^{-1} R_i f_i, \qquad where \qquad R_i \in \mathcal{R},\ f_i \in \overline{F} \tag{2}$$

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A function $f : \mathbb{N} \to \mathbb{N}$ is called a *relative isoperimetric function* of (1) if for any $n \in \mathbb{N}$ and for any word $W \in (X \cup \widetilde{\mathcal{P}})^*$ of length $|W| \leqslant n$ representing the trivial element of the group G, we have

$$Area^{rel}(W) \leqslant f(n).$$

Relative Dehn functions

The smallest relative isoperimetric function of the relative presentation

$$G = \langle X, \mathbb{P} \,|\, \mathcal{R} \rangle. \tag{1}$$

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• For finite relative presentations, δ^{rel} is not always well-defined, i.e. it can be infinite for certain values of the argument:
The group $G = \mathbb{Z} \times \mathbb{Z} = \langle a, b \,|\, [a, b] = 1 \rangle$ has a relative presentation with $X = \{b\}$ and $P = \langle a \rangle$:

$$G = \langle \{b\}, P \,|\, [a, b] = 1 \rangle$$

The word $W_n = [a^n, b]$ has length 4 as a word over $\{b\} \cup P$, but its area equals to n.

Equivalence of Dehn functions

Proposition. Let
$$\langle X_1, (P_\lambda)_{\lambda\in\Lambda} \,|\, \mathcal{R}_1\rangle$$
and
$$\langle X_2, (P_\lambda)_{\lambda\in\Lambda} \,|\, \mathcal{R}_2\rangle$$
be two finite relative presentations of the same group G with respect to a fixed collection of subgroups $(P_\lambda)_{\lambda\in\Lambda}$, and let δ_1 and δ_2 be the corresponding relative Dehn functions. Suppose that δ_1 is well-defined, i.e. δ_1 is finite for every n. Then δ_2 is well-defined and $\delta_1 \sim \delta_2$.

Relatively hyperbolic groups

Definition. (Osin) Let G be a group, $\mathbb{P} = (P_\lambda)_{\lambda \in \Lambda}$ a collection of subgroups of G. The group G is called hyperbolic relative to $\mathbb{P}$, if

(1) G is finitely presented with respect to $\mathbb{P}$ and

(2) The relative Dehn function $\delta^{rel}_{(G,\mathbb{P})}$ is linear.

In this situation we also say that $(G, \mathbb{P})$ is *relatively hyperbolic* and that $\mathbb{P}$ is a *peripheral structure* for G.

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Remark. Conditions (1)&(2) are equivalent to conditions (1)&(3):

(3) The relative Dehn function $\delta^{rel}_{(G,\mathbb{P})}$ is well-defined and

the Cayley graph $\Gamma(G, X \cup \mathcal{P})$ is a hyperbolic metric space.

The main difficulty and the resulting assumption

Difficulty: The space $\Gamma(G, X \cup \mathcal{P})$ is hyperbolic, but is not locally finite if X or $\mathcal{P}$ is infinite.

Assumption. The group G is <u>generated by a finite set X</u> and $(G, \mathbb{P})$ is relatively hyperbolic.

Notation. There are two distance functions on $\Gamma(G, X \cup \mathcal{P})$, $dist_{X \cup \mathcal{P}}$ and $dist_X$. So, we use notation $|AB|_{X \cup \mathcal{P}}$ and $|AB|_X$.

We use blue color to draw geodesic lines with respect to X.

Useful theorem

Theorem. (Osin) For any triple $(G, \mathbb{P}, X)$ satisfying the above assumption, there exists a constant $\nu > 0$ with the following property.

Let Δ be a triangle whose sides p, q, r are geodesics in $\Gamma(G, X \cup \mathcal{P})$. Then for any vertex v on p, there exists a vertex u on the union $q \cup r$ such that

$$dist_X(u, v) < \nu.$$

Parabolic, hyperbolic and loxodromic elements

Let $(G, (P_\lambda)_{\lambda \in \Lambda})$ be relatively hyperbolic. An element $g \in G$ is called

- *parabolic* if it is conjugate into one of the subgroups P_λ, $\lambda \in \Lambda$
- *hyperbolic* if it is not parabolic
- *loxodromic* if it is hyperbolic and has infinite order.

Properties of loxodromic elements

Suppose that $(G, \mathbb{P}, X)$ satisfies the above assumption.

Theorem (Osin) For any loxodromic element $g \in G$, there exist $\lambda > 0$, $\sigma \geqslant 0$ such that for any $n \in \mathbb{Z}$ holds

$$|g^n|_{X \cup \mathcal{P}} \geqslant \lambda |n| - \sigma.$$

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Recall that a subgroup of a group is called *elementary* if it contains a cyclic subgroup of finite index.

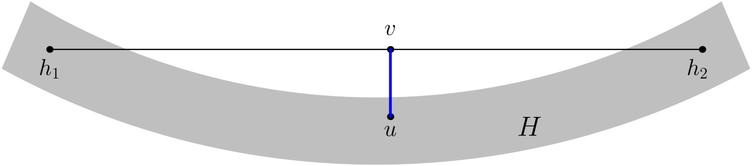
Theorem. (Osin) Every loxodromic element $g \in G$ is contained in a unique maximal elementary subgroup, namely in

$$E_G(g) = \{f \in G \,|\, f^{-1} g^n f = g^{\pm n} \text{ for some } n \in \mathbb{N}\}.$$

Relatively quasiconvex subgroups

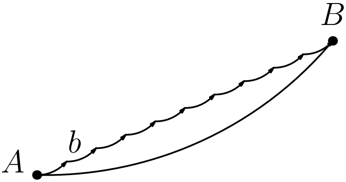
Definition. Let G be a group generated by a finite set X, $\mathbb{P} = \{P_\lambda\}_{\lambda \in \Lambda}$ a collection of subgroups of G.
A subgroup H of G is called relatively quasiconvex with respect to $\mathbb{P}$ if there exists $\epsilon > 0$ such that the following condition holds. Let h_1, h_2 be two elements of H and p an arbitrary geodesic path from h_1 to h_2 in $\Gamma(G, X \cup \mathcal{P})$. Then for any vertex $v \in p$, there exists a vertex $u \in H$ such that

$$dist_X(v, u) \leqslant \epsilon.$$



Else one property of loxodromic elements

Lemma. For every loxodromic element $b \in G$, there exists $\tau > 0$ such that the following holds. Let m be a natural number and $[A, B]$ a geodesic segment in $\Gamma(G, X \cup \mathcal{P})$ connecting 1 and b^m, Then the Hausdorff distance (induced by the $dist_X$-metric) between the sets $[A, B]$ and $\{b^i \mid 0 \leqslant i \leqslant m\}$ is at most τ.



Main theorem

Theorem 1. (BB) Suppose that a finitely generated group G is hyperbolic relative to a collection of subgroups $\mathbb{P} = \{P_1, \ldots, P_m\}$. Let H_1, H_2 be subgroups of G such that

- H_1 is relatively quasiconvex with respect to $\mathbb{P}$ and
- H_2 has a loxodromic element.

Suppose that H_2 is elementwise conjugate into H_1. Then there exists a finite index subgroup of H_2 which is conjugate into H_1.

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The length of the conjugator w.r.t. a finite generating set X of G can be bounded in terms of $|X|$, ϵ_1, $dist_X(1, b)$, where ϵ_1 is a quasi-convexity constant of H_1, and b is a loxodromic element of H_2.

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Remark. Passage to a finite index subgroup of H_2 cannot be avoided:

$$\begin{array}{ccccc} F_2 & \geqslant & H_2 & \geqslant & H_1 \\ \downarrow & & \downarrow & & \downarrow \\ A_4 & \geqslant & K & \geqslant & \mathbb{Z}_2 \end{array}$$

Corollaries

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The index depends only on H_1. The length of the conjugator with respect to a fixed generating system X of G depends only on H_1 and

$$m = \begin{cases} \min\limits_{g \in hyp(H_2)} dist_X(1, g) & \text{if } hyp(H_2) \neq \emptyset, \\ \min\limits_{g \in H_2 \setminus \{1\}} dist_X(1, g) & \text{otherwise.} \end{cases}$$

Here $hyp(H_2)$ denotes the set of hyperbolic elements of H_2.

Corollaries

Definition. (BG) A group G is called subgroup conjugacy separable (abbreviated as SCS) if any two finitely generated and non-conjugate subgroups of G remain non-conjugate in some finite quotient of G. An into-conjugacy version of SCS is abbreviated by SICS.

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Corollary 2. (BB, alternatively Zalesski and Chagas) Limit groups are SICS and SCS.

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First steps of the proof

Take a loxodromic element $b \in H_2$ and an arbitrary $a \in H_2$.
There exists $z_n \in G$ such that $z_n^{-1}(b^n a)z_n \in H_1$:

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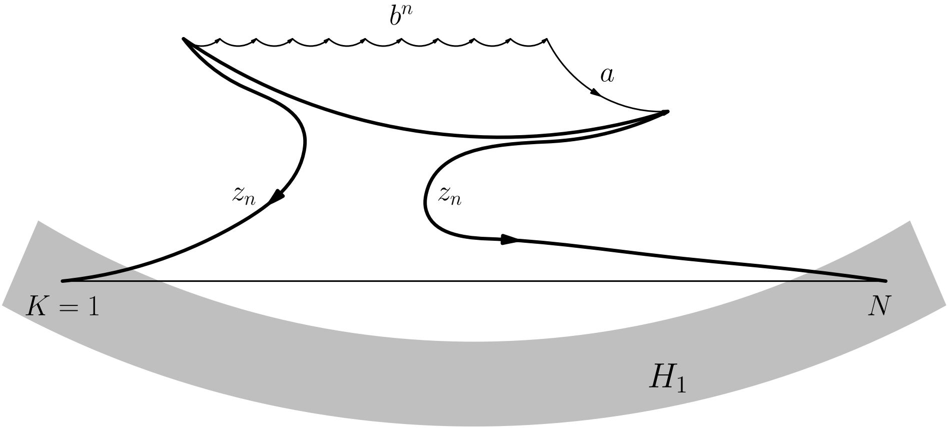
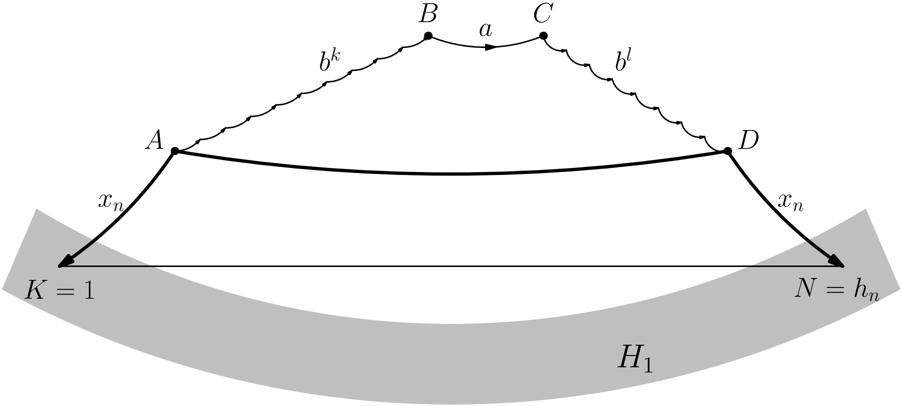


How to avoid large "cancellations" between the blue and red lines?

Change of the conjugator z_n



$$z_n^{-1}(b^n a) z_n = x_n^{-1} \underset{c}{\cdot} (b^k a b^\ell) \underset{c}{\cdot} x_n$$

Change of the conjugator

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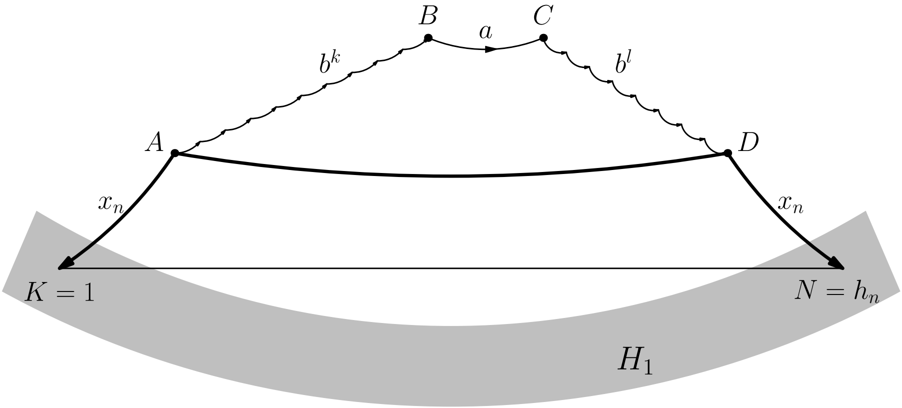
Lemma. Given two elements $a, b \in G$, where b is loxodromic, there exists a constant $c = c(a, b) > 0$ such that for all $n \in \mathbb{N}$ and $z_n \in G$

$$\boxed{z_n^{-1}(b^n a) z_n = x_n^{-1} \underset{c}{\cdot} (b^k a b^\ell) \underset{c}{\cdot} x_n}$$

for some $x_n \in G$ and $k, \ell \in \mathbb{N}$ with $n = k + \ell$.

Proof of Theorem



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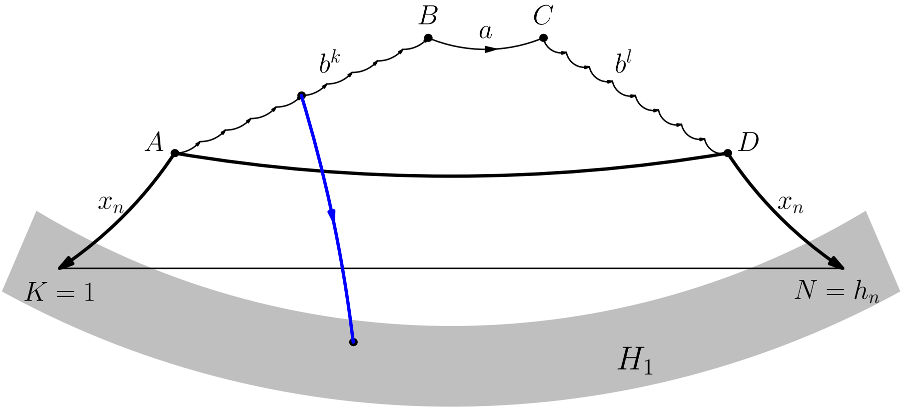

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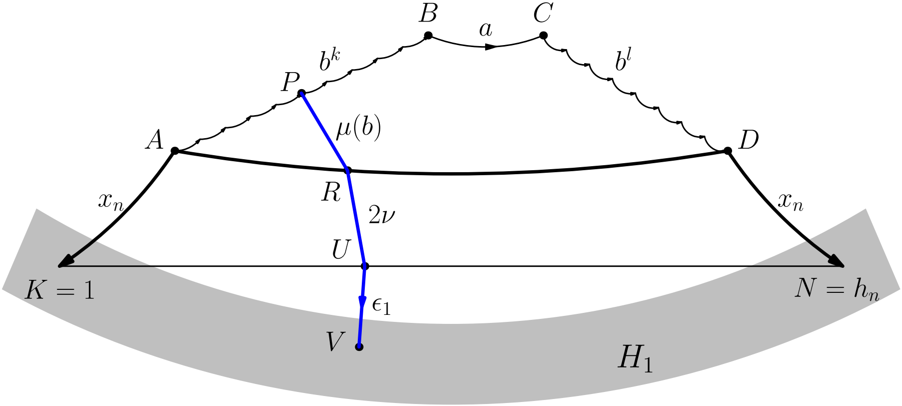
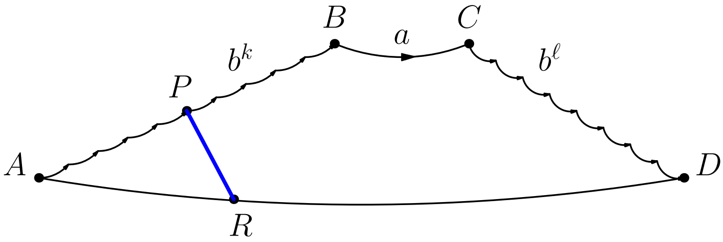


Lemma 1



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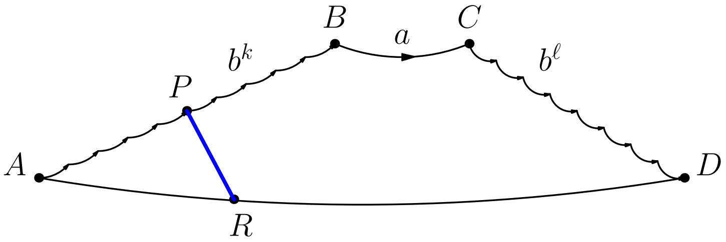
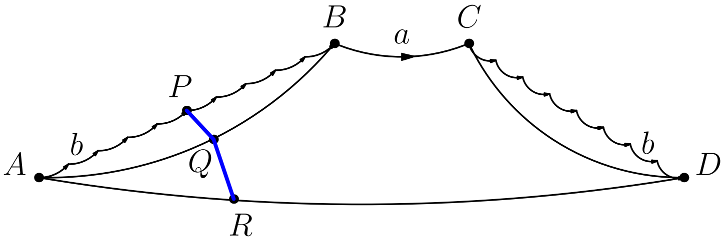


For all sufficiently large k and every vertex P in the middle third of the waved line AB, there exists a vertex $R \in [A, D]$ such that

$$dist_X(P, R) < \mu(b).$$

Proof of Lemma 1



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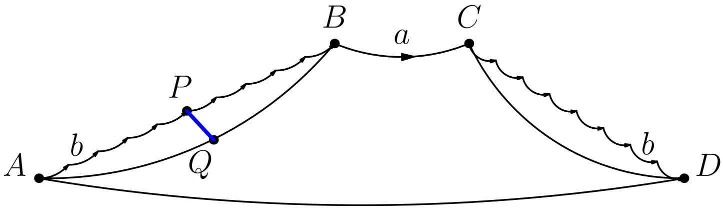

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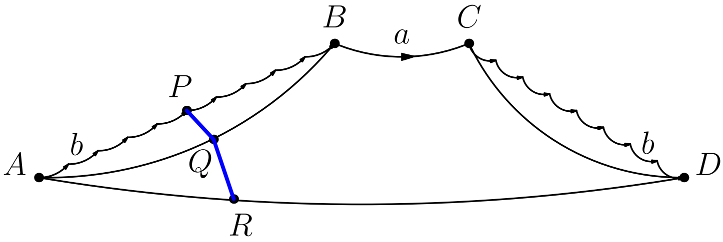

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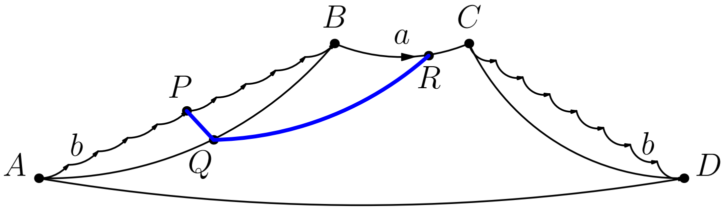

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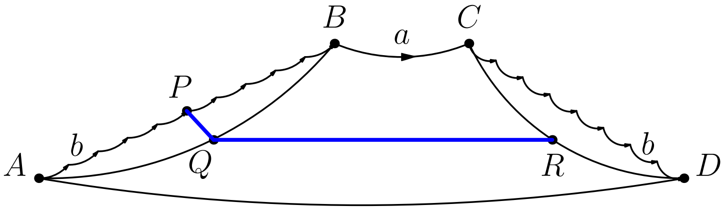

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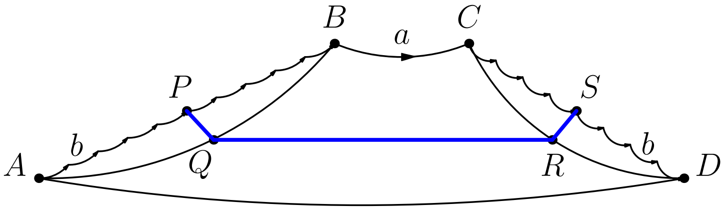

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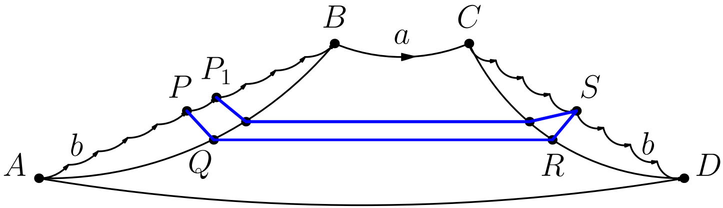

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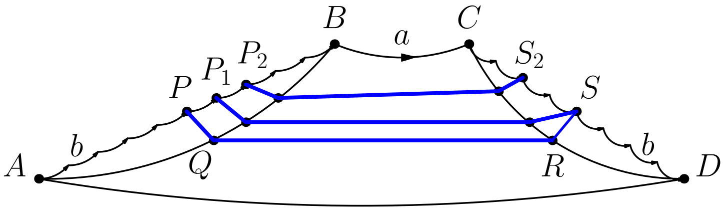

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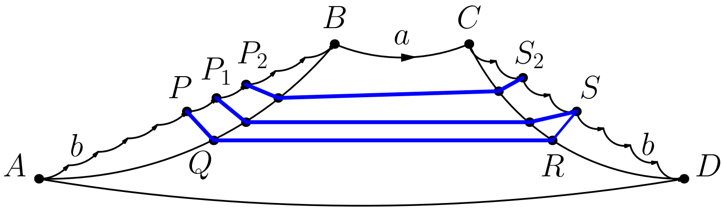

$$Label([P_iS_i]) \underset{G}{=} b^{k_i}ab^{l_i}.$$

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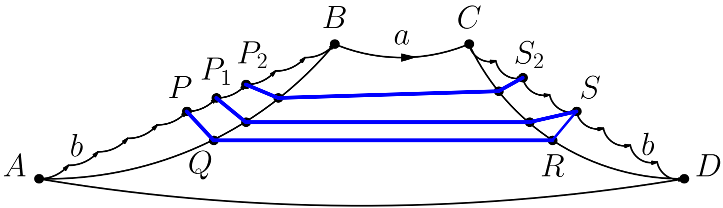

$Label([P_iS_i]) \underset{G}{=} b^{k_i}ab^{l_i}$.

Repetition of labels: $b^{k_i}ab^{l_i} = b^{k_j}ab^{l_j}$

$a^{-1}b^{k_i-k_j}a = b^{l_j-l_i}$

Hence $a \in E_G(b)$, a contradiction.

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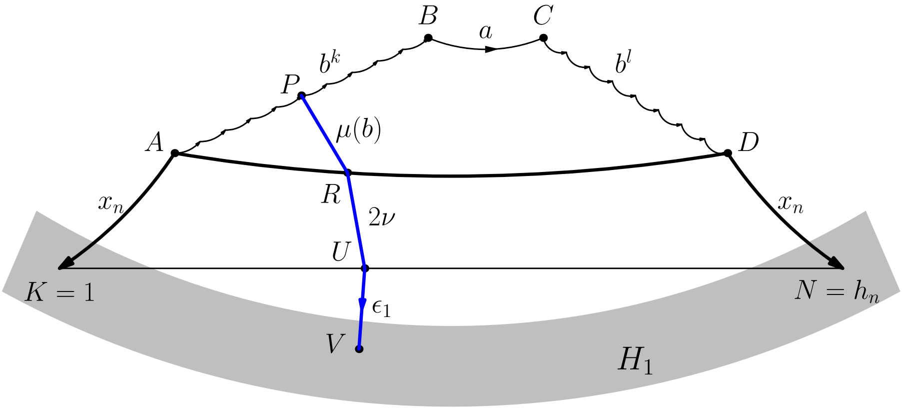

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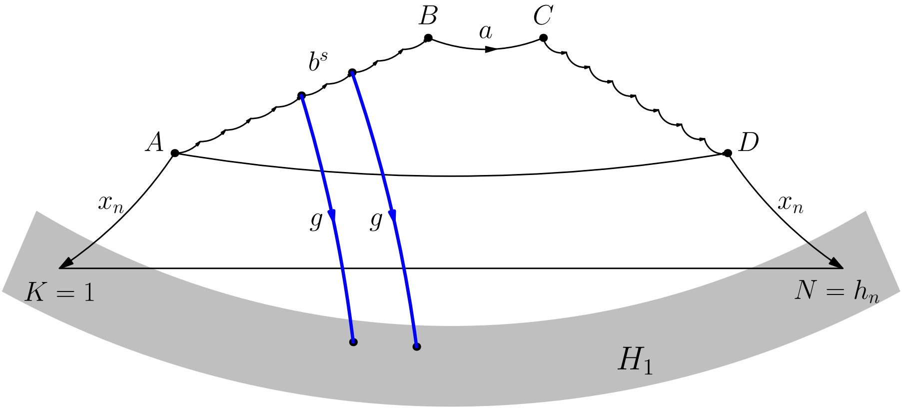

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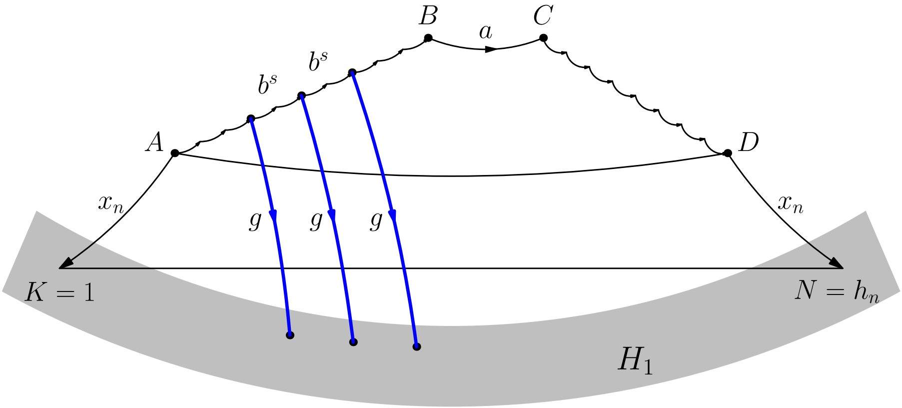

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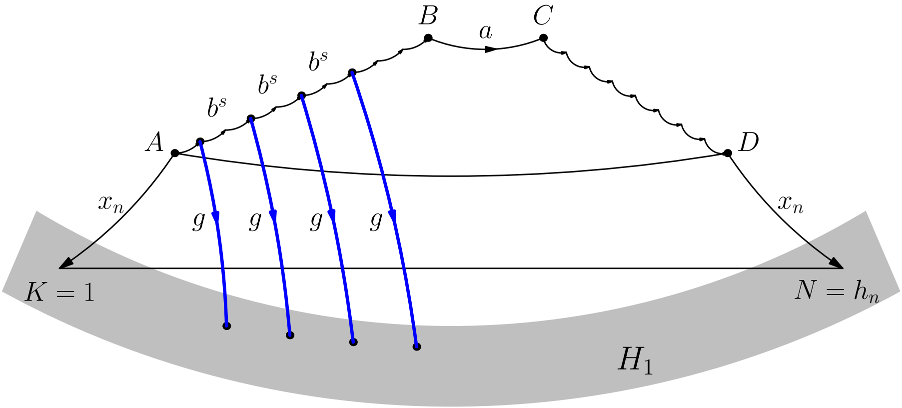

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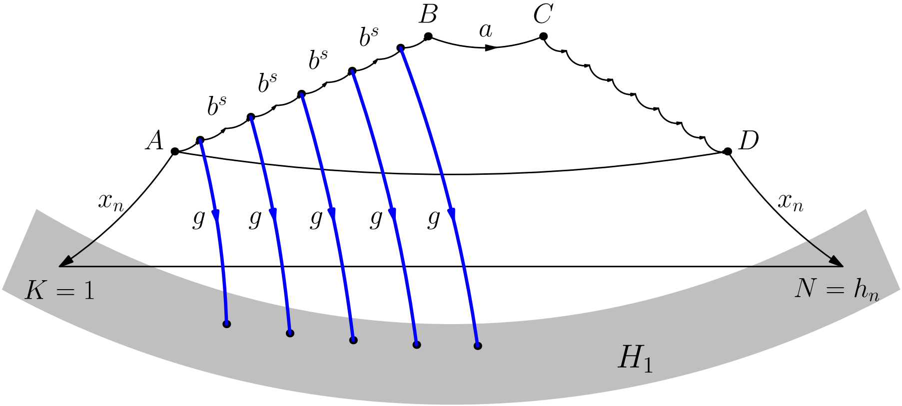

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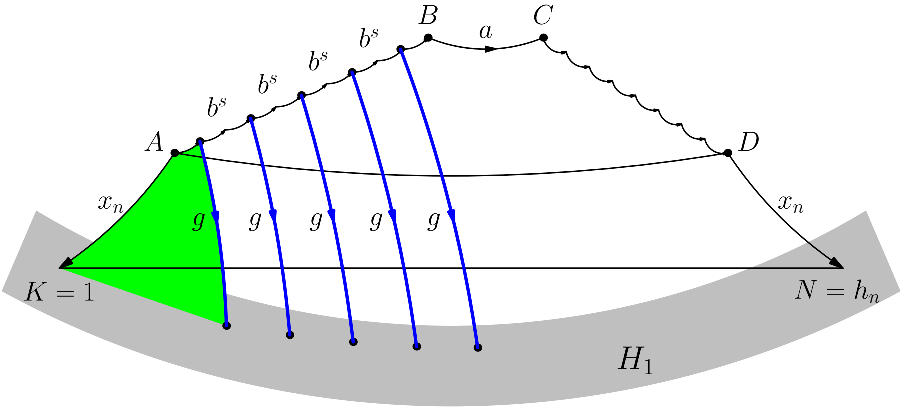

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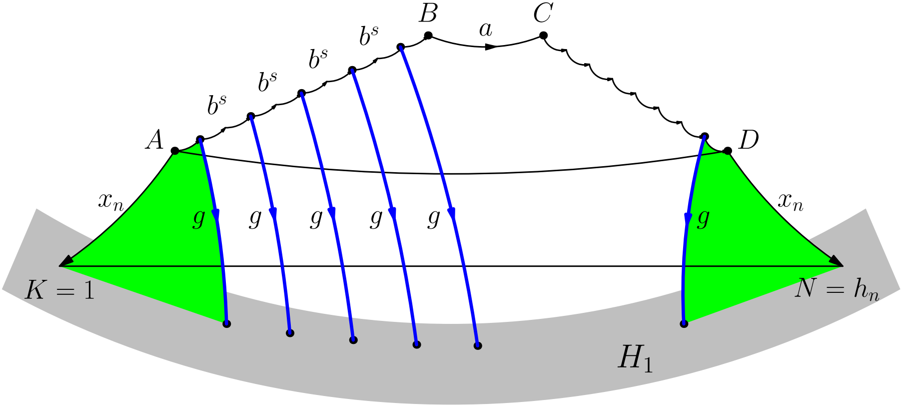

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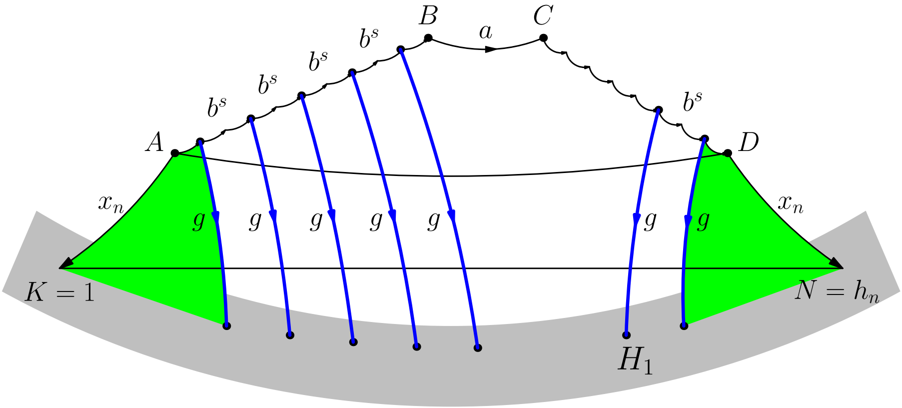

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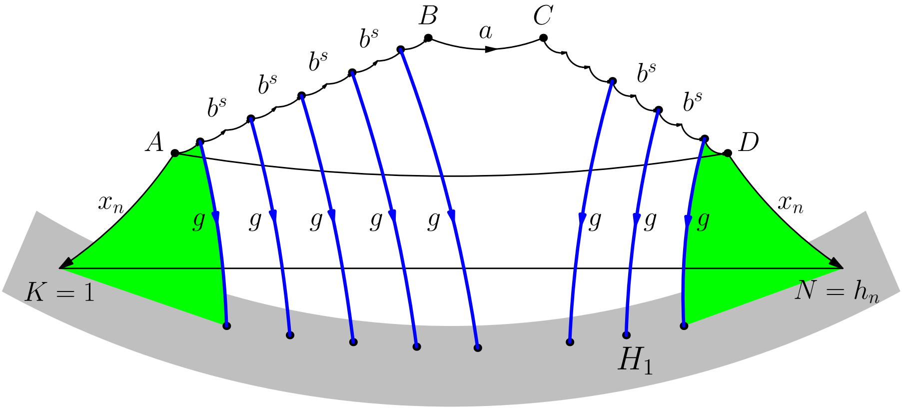

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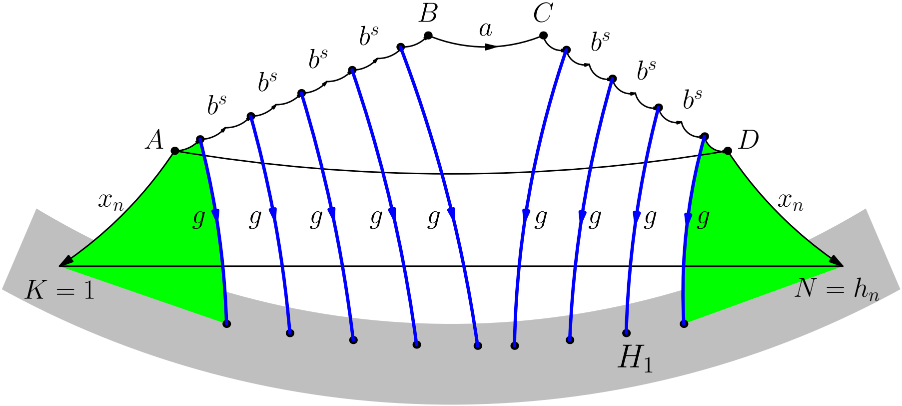

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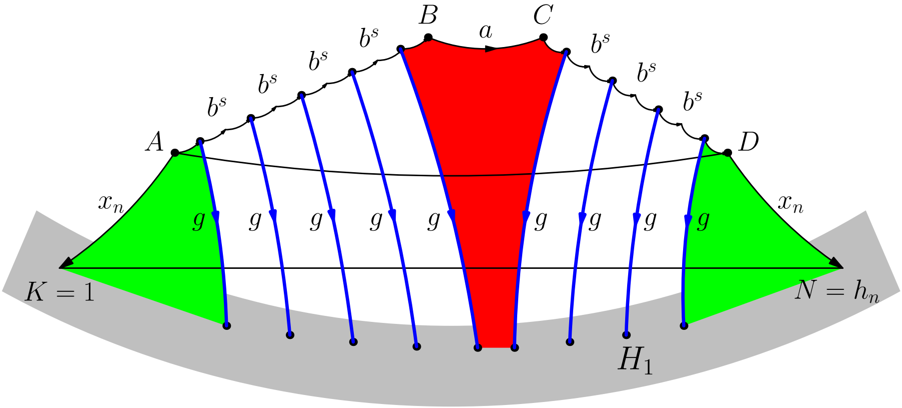

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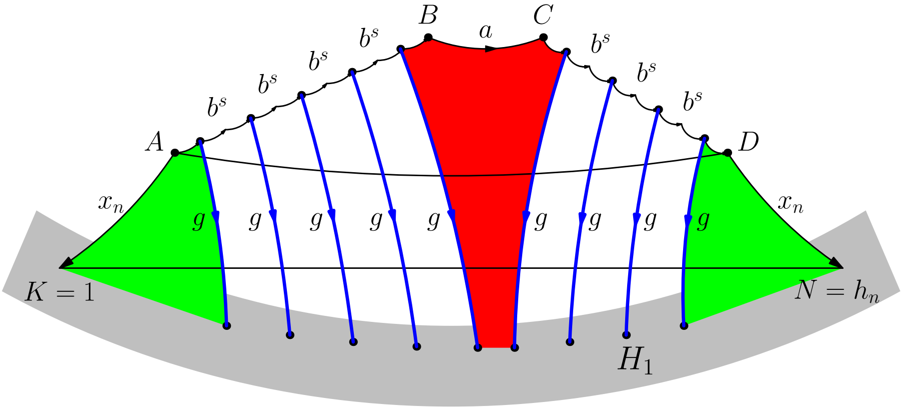

$g^{-1} b^p a b^q g \in H_1,$ $\qquad |g|_X \leqslant f_1(b),\ 0 \leqslant p, q < s \leqslant f_2(b)$

Proof of Theorem

$$g^{-1}b^p a b^q g \in H_1, \qquad \text{where } |g|_X, p, q \text{ are bounded in terms of } b.$$

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$$a \in z^{-1}H_1 z \cdot b^t, \qquad \text{where } |z|_X \text{ and } t \text{ are bounded in terms of } b.$$

Proof of Theorem

$g^{-1}b^p a b^q g \in H_1,$ where $|g|_X, p, q$ are bounded in terms of b.

$a \in z^{-1}H_1 z \cdot b^t,$ where $|z|_X$ and t are bounded in terms of b.

$$H_2 \subseteq \bigcup_{(z,t)\in M} z^{-1}H_1 z \cdot b^t \bigcup E_G(b).$$

$$H_2 = \bigcup_{(z,t)\in M} (z^{-1}H_1 z \cap H_2) \cdot b^t \bigcup (E_G(b) \cap H_2).$$

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Theorem. (B.H. Neumann) If a group G is covered by a finite number of some cosets of subgroups of G, then among these subgroups, there is a subgroup of finite index in G.

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Theorem. (B.H. Neumann) If a group G is covered by a finite number of some cosets of subgroups of G, then among these subgroups, there is a subgroup of finite index in G.

Thus, one of the following subgroups has finite index in H_2:

- $z^{-1}H_1 z \cap H_2$
- $E_G(b) \cap H_2$

THANK YOU!