From local to global conjugacy in relatively hyperbolic groups

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Relative presentations

Let $G$ be a group, $\mathbb{P} = \{P_\lambda\}_{\lambda \in \Lambda}$ a collection of subgroups of $G$, $X$ a subset of $G$. We say that $X$ is a relative generating set of $G$ with respect to $\mathbb{P}$ if

$$G = \langle \bigcup_{\lambda \in \Lambda} P_\lambda \cup X \rangle.$$ 

In this situation $G$ can be regarded as a quotient group of

$$\overline{F} = \bigstar_{\lambda \in \Lambda} \tilde{P}_\lambda \ast F(X),$$

where $\tilde{P}_\lambda$ is a copy of $P_\lambda$ such that the union of all $\tilde{P}_\lambda \setminus \{1\}$ and $X$ is disjoint.
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where $\tilde{P}_\lambda$ is a copy of $P_\lambda$ such that the union of all $\tilde{P}_\lambda \setminus \{1\}$ and $X$ is disjoint. We will use the most useless presentation of $\tilde{P}_\lambda$:

$$\tilde{P}_\lambda = \langle \tilde{P}_\lambda \setminus \{1\} | \tilde{S}_\lambda \rangle,$$

where $\tilde{S}_\lambda$ is the set of all words over the alphabet $\tilde{P}_\lambda \setminus \{1\}$ that represent 1 in the group $\tilde{P}_\lambda$. Denote

$$\tilde{\mathcal{P}} = \bigcup_{\lambda \in \Lambda} (\tilde{P}_\lambda \setminus \{1\}), \quad \tilde{\mathcal{S}} := \bigcup_{\lambda \in \Lambda} \tilde{S}_\lambda.$$
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We will use the most useless presentation $\tilde{P}_\lambda = \langle \tilde{P}_\lambda \setminus \{1\} \mid \tilde{S}_\lambda \rangle$, and the sets $\tilde{\mathcal{P}}$ and $\tilde{S}$ as above. Then $\overline{F}$ has the presentation

$$\overline{F} = \langle \tilde{\mathcal{P}} \sqcup X \mid \tilde{S} \rangle$$

and $G$ has a presentation (called relative with respect to $\mathbb{P}$)

$$G = \langle \tilde{\mathcal{P}} \sqcup X \mid \tilde{S} \sqcup \mathcal{R} \rangle.$$
Finite relative presentations

The relative presentation

\[ G = \langle \tilde{P} \sqcup X \mid \tilde{S} \sqcup R \rangle \]

can be briefly written as

\[ G = \langle X, \mathbb{P} \mid \mathcal{R} \rangle. \]

This relative presentation is called finite if \( X \) and \( \mathcal{R} \) are finite.
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can be briefly written as

\[ G = \langle X, P \mid R \rangle. \]

This relative presentation is called finite if \( X \) and \( R \) are finite.

**Example.** Consider the amalgamated product

\[ G = H_1 \ast_{K}^{\alpha} H_2. \]

With \( P = \{ H_1, H_2 \} \), there is the following relative presentation

\[ G = \langle \emptyset, P \mid k = \alpha(k) \ (k \in K) \rangle. \]

It can be chosen finite if \( K \) is finitely generated.
Relative isoperimetric functions

Suppose that $G$ has a relative presentation

$$G = \langle X, (P_\lambda)_{\lambda \in \Lambda} \mid R \rangle. \quad (1)$$

Then $G$ is a quotient of

$$\overline{F} = \left( \bigast_{\lambda \in \Lambda} \tilde{P}_\lambda \right) \ast F(X)$$

If a word $W \in (X \cup \tilde{P})^*$ represents 1 in $G$, there exists an expression

$$W = \overline{F} \prod_{i=1}^{k} f_i^{-1} R_i f_i, \quad \text{where} \quad R_i \in R, \ f_i \in \overline{F} \quad (2)$$

The smallest possible number $k$ in a representation of type (2) is denoted $\text{Area}_{rel}(W)$. 
Relative isoperimetric functions

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If a word $W \in (X \cup \tilde{P})^*$ represents 1 in $G$, there exists an expression

$$W \equiv \prod_{i=1}^{k} f_i^{-1} R_i f_i, \quad \text{where} \quad R_i \in R, \ f_i \in \overline{F} \quad (2)$$

The smallest possible number $k$ in a representation of type (2) is denoted $Area^{rel}(W)$.

A function $f : \mathbb{N} \to \mathbb{N}$ is called a relative isoperimetric function of (1) if for any $n \in \mathbb{N}$ and for any word $W \in (X \cup \tilde{P})^*$ of length $|W| \leq n$ representing the trivial element of the group $G$, we have

$$Area^{rel}(W) \leq f(n).$$
Relative Dehn functions

The smallest relative isoperimetric function of the relative presentation

\[ G = \langle X, \mathbb{P} \mid \mathcal{R} \rangle. \] (1)

is called the relative Dehn function of \( G \) with respect to \( \{ P_\lambda \}_{\lambda \in \Lambda} \) and is denoted by \( \delta_{\text{rel}} \left( G, \mathbb{P} \right) \).
Relative Dehn functions

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- For finite relative presentations, \( \delta^{\text{rel}} \) is not always well-defined, i.e. it can be infinite for certain values of the argument:

The group \( G = \mathbb{Z} \times \mathbb{Z} = \langle a, b | [a, b] = 1 \rangle \) has a relative presentation with \( X = \{ b \} \) and \( P = \langle a \rangle \):

\[ G = \langle \{ b \}, P | [a, b] = 1 \rangle \]

The word \( W_n = [a^n, b] \) has length 4 as a word over \( \{ b \} \cup P \), but its area equals to \( n \).
Equivalence of Dehn functions

Proposition. Let

\[ \langle X_1, (P_\lambda)_{\lambda \in \Lambda} | R_1 \rangle \]

and

\[ \langle X_2, (P_\lambda)_{\lambda \in \Lambda} | R_2 \rangle \]

be two finite relative presentations of the same group \( G \) with respect to a fixed collection of subgroups \( (P_\lambda)_{\lambda \in \Lambda} \), and let \( \delta_1 \) and \( \delta_2 \) be the corresponding relative Dehn functions. Suppose that \( \delta_1 \) is well-defined, i.e. \( \delta_1 \) is finite for every \( n \). Then \( \delta_2 \) is well-defined and \( \delta_1 \sim \delta_2 \).
Relatively hyperbolic groups

**Definition.** (Osin) Let $G$ be a group, $\mathbb{P} = (P_\lambda)_{\lambda \in \Lambda}$ a collection of subgroups of $G$. The group $G$ is called **hyperbolic relative to** $\mathbb{P}$, if

1. $G$ is finitely presented with respect to $\mathbb{P}$ and
2. The relative Dehn function $\delta_{G,\mathbb{P}}$ is linear.

In this situation we also say that $(G, \mathbb{P})$ is **relatively hyperbolic** and that $\mathbb{P}$ is a **peripheral structure** for $G$. 
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**Remark.** Conditions (1)\&(2) are equivalent to conditions (1)\&(3):

3. The relative Dehn function $\delta^\text{rel}_{(G,\mathcal{P})}$ is well-defined and the Cayley graph $\Gamma(G, X \cup \mathcal{P})$ is a hyperbolic metric space.
The main difficulty and the resulting assumption

**Difficulty:** The space $\Gamma(G, X \cup P)$ is hyperbolic, but is not locally finite if $X$ or $P$ is infinite.

**Assumption.** The group $G$ is generated by a finite set $X$ and $(G, P)$ is relatively hyperbolic.

**Notation.** There are two distance functions on $\Gamma(G, X \cup P)$, $\text{dist}_{X \cup P}$ and $\text{dist}_X$. So, we use notation $|AB|_{X \cup P}$ and $|AB|_X$. We use **blue** color to draw geodesic lines with respect to $X$. 
Theorem. (Osin) For any triple \((G, \mathbb{P}, X)\) satisfying the above assumption, there exists a constant \(\nu > 0\) with the following property.

Let \(\Delta\) be a triangle whose sides \(p, q, r\) are geodesics in \(\Gamma(G, X \cup \mathbb{P})\). Then for any vertex \(v\) on \(p\), there exists a vertex \(u\) on the union \(q \cup r\) such that

\[
dist_X(u, v) < \nu.
\]
Parabolic, hyperbolic and loxodromic elements

Let \((G, (P_{\lambda})_{\lambda \in \Lambda})\) be relatively hyperbolic. An element \(g \in G\) is called

- **parabolic** if it is conjugate into one of the subgroups \(P_{\lambda}, \lambda \in \Lambda\)
- **hyperbolic** if it is not parabolic
- **loxodromic** if it is hyperbolic and has infinite order.
Properties of loxodromic elements

Suppose that \((G, \mathbb{P}, X)\) satisfies the above assumption.

**Theorem** (Osin) For any loxodromic element \(g \in G\), there exist \(\lambda > 0, \sigma \geq 0\) such that for any \(n \in \mathbb{Z}\) holds

\[ |g^n|_{X \cup \mathbb{P}} \geq \lambda |n| - \sigma. \]
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\]

Recall that a subgroup of a group is called *elementary* if it contains a cyclic subgroup of finite index.

**Theorem. (Osin)** Every loxodromic element \(g \in G\) is contained in a unique maximal elementary subgroup, namely in

\[
E_G(g) = \{f \in G \mid f^{-1}g^n f = g^{\pm n} \text{ for some } n \in \mathbb{N}\}.
\]
Relatively quasiconvex subgroups

Definition. Let $G$ be a group generated by a finite set $X$, $\mathbb{P} = \{P_\lambda\}_{\lambda \in \Lambda}$ a collection of subgroups of $G$. A subgroup $H$ of $G$ is called relatively quasiconvex with respect to $\mathbb{P}$ if there exists $\epsilon > 0$ such that the following condition holds. Let $h_1, h_2$ be two elements of $H$ and $p$ an arbitrary geodesic path from $h_1$ to $h_2$ in $\Gamma(G, X \cup \mathbb{P})$. Then for any vertex $v \in p$, there exists a vertex $u \in H$ such that

$$\text{dist}_X(v, u) \leq \epsilon.$$
Lemma. For every loxodromic element $b \in G$, there exists $\tau > 0$ such that the following holds. Let $m$ be a natural number and $[A, B]$ a geodesic segment in $\Gamma(G, X \cup \mathcal{P})$ connecting $1$ and $b^m$. Then the Hausdorff distance (induced by the $dist_X$-metric) between the sets $[A, B]$ and $\{b^i \mid 0 \leq i \leq m\}$ is at most $\tau$. 
Main theorem

Theorem 1. (BB) Suppose that a finitely generated group $G$ is hyperbolic relative to a collection of subgroups $\mathbb{P} = \{P_1, \ldots, P_m\}$. Let $H_1, H_2$ be subgroups of $G$ such that

- $H_1$ is relatively quasiconvex with respect to $\mathbb{P}$ and
- $H_2$ has a loxodromic element.

Suppose that $H_2$ is elementwise conjugate into $H_1$. Then there exists a finite index subgroup of $H_2$ which is conjugate into $H_1$.
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Suppose that $H_2$ is elementwise conjugate into $H_1$. Then there exists a finite index subgroup of $H_2$ which is conjugate into $H_1$.

The length of the conjugator w.r.t. a finite generating set $X$ of $G$ can be bounded in terms of $|X|, \epsilon_1, \text{dist}_X(1, b)$, where $\epsilon_1$ is a quasi-convexity constant of $H_1$, and $b$ is a loxodromic element of $H_2$. 
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**Remark.** Passage to a finite index subgroup of $H_2$ cannot be avoided:

\[
\begin{align*}
F_2 & \supseteq H_2 & \supseteq H_1 \\
\downarrow & \quad \downarrow & \quad \downarrow \\
A_4 & \supseteq K & \supseteq \mathbb{Z}_2
\end{align*}
\]
Corollaries

**Theorem.** (Dahmani and, alternatively Alibegović) Limit groups are hyperbolic relative to a collection of representatives of conjugacy classes of maximal noncyclic abelian subgroups.

Corollary 1. Let $G$ be a limit group and let $H_1$ and $H_2$ be subgroups of $G$, where $H_1$ is finitely generated. Suppose that $H_2$ is elementwise conjugate into $H_1$. Then there exists a finite index subgroup of $H_2$ which is conjugate into $H_1$. The index depends only on $H_1$. The length of the conjugator with respect to a fixed generating system $X$ of $G$ depends only on $H_1$ and $m = \begin{cases} \min_{g \in \text{hyp}(H_2)} \text{dist}(1, g) & \text{if } \text{hyp}(H_2) \neq \emptyset, \\ \min_{g \in H_2 \setminus \{1\}} \text{dist}(1, g) & \text{otherwise}. \end{cases}$ Here $\text{hyp}(H_2)$ denotes the set of hyperbolic elements of $H_2$. 
Corollaries

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$$m = \begin{cases} 
\min_{g \in \text{hyp}(H_2)} \text{dist}_X(1, g) & \text{if } \text{hyp}(H_2) \neq \emptyset, \\
\min_{g \in H_2 \setminus \{1\}} \text{dist}_X(1, g) & \text{otherwise}.
\end{cases}$$

Here $\text{hyp}(H_2)$ denotes the set of hyperbolic elements of $H_2$. 
Definition. (BG) A group $G$ is called \textbf{subgroup conjugacy separable} (abbreviated as SCS) if any two finitely generated and non-conjugate subgroups of $G$ remain non-conjugate in some finite quotient of $G$. An into-conjugacy version of SCS is abbreviated by SICS.
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**Definition.** (BG) A group $G$ is called **subgroup conjugacy separable** (abbreviated as SCS) if any two finitely generated and non-conjugate subgroups of $G$ remain non-conjugate in some finite quotient of $G$. An into-conjugacy version of SCS is abbreviated by SICS.

**Corollary 2.** (BB, alternatively Zalesski and Chagas) Limit groups are SICS and SCS.
Theorem 1. (BB) Suppose that a finitely generated group $G$ is hyperbolic relative to a collection of subgroups $\mathbb{P} = \{P_1, \ldots, P_m\}$. Let $H_1, H_2$ be subgroups of $G$ such that

- $H_1$ is relatively quasiconvex with respect to $\mathbb{P}$ and
- $H_2$ has a loxodromic element.

Suppose that $H_2$ is elementwise conjugate into $H_1$. Then there exists a finite index subgroup of $H_2$ which is conjugate into $H_1$.

The length of the conjugator w.r.t. a finite generating set $X$ of $G$ can be bounded in terms of $|X|, \epsilon_1, dist_X(1, b)$, where $\epsilon_1$ is a quasiconvexity constant of $H_1$, and $b$ is a loxodromic element of $H_2$. 
First steps of the proof

Take a loxodromic element $b \in H_2$ and an arbitrary $a \in H_2$. There exists $z_n \in G$ such that $z_n^{-1}(b^n a)z_n \in H_1$: 
First steps of the proof

Take a loxodromic element \( b \in H_2 \) and an arbitrary \( a \in H_2 \). There exists \( z_n \in G \) such that \( z_n^{-1}(b^n a)z_n \in H_1 \):

How to avoid large “cancellations” between the blue and red lines?
Change of the conjugator $z_n$

\[
z_n^{-1}(b^n a)z_n = x_n^{-1} c (b^k ab^\ell) c x_n
\]
Change of the conjugator

Notation: For $u, v \in G$ and $c > 0$, we write $u \cdot_c v$ if

$$|uv| \geq |u| + |v| - 2c.$$
**Change of the conjugator**

**Notation:** For $u, v \in G$ and $c > 0$, we write $u \cdot_c v$ if

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**Lemma.** Given two elements $a, b \in G$, where $b$ is loxodromic, there exists a constant $c = c(a, b) > 0$ such that for all $n \in \mathbb{N}$ and $z_n \in G$

$$z_n^{-1}(b^n a)z_n = x_n^{-1}_{c} \cdot (b^k ab^\ell)_{c} \cdot x_n$$

for some $x_n \in G$ and $k, \ell \in \mathbb{N}$ with $n = k + \ell$. 
Proof of Theorem

\[ \begin{align*}
K &= 1 \\
n &= \frac{1}{N} \\
A &= x_n \\
B &= b^k \\
C &= a \\
D &= b^l \\
E &= x_n \\
F &= N = h_n
\end{align*} \]
Proof of Theorem

\[ a^n b^k b^l = 1 \]

\[ N = h_n \]

\[ K = 1 \]
Proof of Theorem
For all sufficiently large $k$ and every vertex $P$ in the middle third of the waved line $AB$, there exists a vertex $R \in [A, D]$ such that $\text{dist}_X(P, R) < \mu(b)$.
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Proof of Lemma 1
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Let \( P_i \) and \( S_i \) be the relevant labels. Then

\[
\begin{align*}
\text{Label} & (P_i S_i) = G \quad b_k a b_k i = b_k j a b_k j = a - 1 b_k i - k j = b_l j - l i.
\end{align*}
\]

Hence \( a \in E \) \( G \), a contradiction.
Proof of Lemma 1

Label\left([P_iS_i]\right) = b^{k_i}a^{l_i}. 
Proof of Lemma 1

Label([P_i; S_i]) = \frac{b^{k_i} a b^{l_i}}{G}.

Repetition of labels: \ b^{k_i} a b^{l_i} = b^{k_j} a b^{l_j}
\ a^{−1} b^{k_i−k_j} a = b^{l_j−l_i}
Hence \ a \in E_G(b), \ a \ contradiction.
Proof of Theorem
Proof of Theorem

\[ \text{Diagram:}\]

- Points: A, B, C, D
- Lines: AB, BC, AC, BD
- Arrows: A → B, B → C
- Points labeled: \( x_n \), \( b^s \), \( g \), \( g \)
- Representations: \( K = 1 \), \( N = h_n \), \( H_1 \)

\[ \text{Text:}\]

- \( A \) to \( B \) to \( C \) to \( D \)
- \( x_n \) at each point
- \( b^s \) and \( g \) in diagram

\[ \text{Math:}\]

- \( K = 1 \)
- \( N = h_n \)
- \( H_1 \)
Proof of Theorem
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\[ K = 1 \quad N = h_n \]

\[ H_1 \]
Proof of Theorem
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\[ K = 1 \quad N = h_n \]

\[ A \quad B \quad C \quad D \]

\[ x_n \quad g \quad g \quad g \quad g \quad g \quad x_n \]

\[ H_1 \]
Proof of Theorem
Proof of Theorem

$$K = 1, N = h_n$$

$$A, B, C, D$$

$$x_n, A, B, C, D, H_1$$

$$g, g, g, g, g, g, g$$

$$b^s, b^s, b^s, b^s, b^s, b^s, b^s$$

$$a, C$$
Proof of Theorem

\[ K = 1 \quad N = h_n \]
Proof of Theorem

\[ K = 1 \]

\[ N = h_n \]

\[ H_1 \]
Proof of Theorem

\[ g^{-1} b^p a b^q g \in H_1, \quad |g| \chi \leq f_1(b), \quad 0 \leq p, q < s \leq f_2(b) \]
Proof of Theorem

\[ g^{-1} b^p a b^q g \in H_1, \]

where \(|g|_X, p, q\) are bounded in terms of \(b\).
Proof of Theorem

\[ g^{-1} b^p a b^q g \in H_1, \quad \text{where } |g|_X, p, q \text{ are bounded in terms of } b. \]

\[ a \in z^{-1} H_1 z \cdot b^t, \quad \text{where } |z|_X \text{ and } t \text{ are bounded in terms of } b. \]
Proof of Theorem

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\[ H_2 \subseteq \bigcup_{(z,t)\in M} z^{-1}H_1z \cdot b^t \cup E_G(b). \]

\[ H_2 = \bigcup_{(z,t)\in M} (z^{-1}H_1z \cap H_2) \cdot b^t \cup (E_G(b) \cap H_2). \]
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**Theorem.** (B.H. Neumann) If a group \( G \) is covered by a finite number of some cosets of subgroups of \( G \), then among these subgroups, there is a subgroup of finite index in \( G \).
Proof of Theorem

\[ g^{-1} b^p a b^q g \in H_1, \quad \text{where} \ |g|_X, p, q \text{ are bounded in terms of} \ b. \]

\[ a \in z^{-1} H_1 z \cdot b^t, \quad \text{where} \ |z|_X \text{ and} \ t \text{ are bounded in terms of} \ b. \]

\[ H_2 \subseteq \bigcup_{(z,t)\in M} z^{-1} H_1 z \cdot b^t \cup E_G(b). \]

\[ H_2 = \bigcup_{(z,t)\in M} (z^{-1} H_1 z \cap H_2) \cdot b^t \cup (E_G(b) \cap H_2). \]

**Theorem.** (B.H. Neumann) If a group \( G \) is covered by a finite number of some cosets of subgroups of \( G \), then among these subgroups, there is a subgroup of finite index in \( G \).

Thus, one of the following subgroups has finite index in \( H_2 \):
- \( z^{-1} H_1 z \cap H_2 \)
- \( E_G(b) \cap H_2 \)
THANK YOU!