On spectra of Koopman, groupoid and quasi-regular representations

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Preliminaries

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- Denote by $\sigma(A)$ the spectrum of an operator A.
- Let (X, μ) be a standard Borel space with a measure-class preserving action of a group G on it.

In the talk we compare spectra and spectral measures of operators of representations associated to (G, X, μ).

Weak containment of representations

Definition

Let ρ and η be two unitary representations of a group G acting in Hilbert spaces $\mathcal{H}_\rho$ and $\mathcal{H}_\eta$ correspondingly. Then ρ is weakly contained in η (denoted by $\rho \prec \eta$) if for any $\epsilon > 0$, any finite subset $S \subset G$ and any vector $v \in \mathcal{H}_\rho$ there exists a finite collection of vectors $w_1, \ldots, w_n \in \mathcal{H}_\eta$ such that

$$|(\rho(g)v, v) - \sum_{i=1}^{n}(\eta(g)w_i, w_i)| < \epsilon$$

for all $g \in S$.

Equivalent formulations

For a representation π of G let C_π be the C^* algebra generated by $\pi(g), g \in G$. Results of Dixmier imply:

Proposition

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3) $\|\rho(m)\| \leqslant \|\eta(m)\|$ *for every positive* $m \in \mathbb{C}[G]$ *(i.e.* $m = x^*x$*).*

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3) $\|\rho(m)\| \leqslant \|\eta(m)\|$ *for every positive* $m \in \mathbb{C}[G]$ *(i.e.* $m = x^* x$*).*
4) *there exists a surjective homomorphism* $\phi : C_\eta \to C_\rho$ *such that* $\phi(\eta(g)) = \rho(g)$ *for all* $g \in G$.

Quasi-regular representations

For $x \in X$ let Gx be the orbit of x. The corresponding quasi-regular representation
$\rho_x : G \to U(l^2(Gx))$ is defined by:
$(\rho_x(g)f)(y) = f(g^{-1}y), \ \ f \in l^2(Gx)$.

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Important cases:
$X = G/H, H < G, \rho_{G/H}$.
$X = G, \rho_G$ is the regular representation.

Quasi-regular representations and amenability

Theorem (Kesten)

For symmetric generating measure ν on G and $H \lhd G$ one has

$$\|\rho_{G/H}(\nu)\| = \|\rho_G(\nu)\|$$

if and only if H is amenable.

Quasi-regular representations and amenability

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Proposition

Let $H < G$. Then $\rho_{G/H} \prec \rho_G$ if and only if H is amenable.

Koopman representation

Koopman representation is the representation
$\kappa : G \to U(L^2(X,\mu))$ by

$$(\kappa(g)f)(y) = \sqrt{\frac{\mathrm{d}\mu(g^{-1}y)}{\mathrm{d}\mu(y)}} f(g^{-1}y), \quad f \in L^2(X,\mu).$$

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Ergodicity, weak mixing and mixing can be formulated in terms of spectral properties of κ.

Groupoid representation

- Groupoid representation π is the direct integral of quasi-regular representations

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• Let

$$\mathcal{R} = \{(x, y) : x = gy \ \text{ for some } \ g \in G\} \subset X \times X$$

be the equivalence relation on X generated by the action of G. G acts on $\mathcal{R}$ by $g : (x, y) \to (gx, y)$. There exists a unique G-invariant measure ν on $\mathcal{R}$ such that

$$\nu(\{(x, x) : x \in A\} = \mu(A)$$

for any measurable subset $A \subset X$.

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When μ is G-invariant π is related to character theory, finite factor-representations, non-free actions etc.

Spherically homogeneous rooted tree

For a sequence $\overline{d} = \{d_n\}_{n\in\mathbb{N}}$, $d_n \geqslant 2$ the spherically homogeneous rooted tree $T_{\overline{d}}$ is a tree such that:

• the vertex set $V = \cup_{n\in\mathbb{Z}_+} V_n$;

• $V_0 = \{v_0\}$ where v_0 is called the root;

• each vertex from V_n is connected by an edge to the same number d_n of vertices from V_{n+1}.

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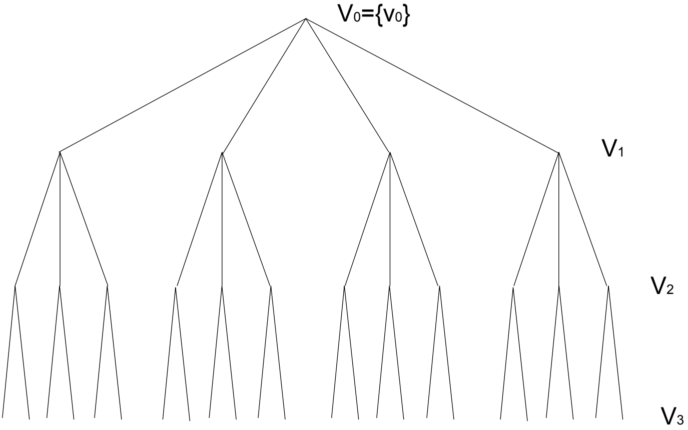
• each vertex from V_n is connected by an edge to the same number d_n of vertices from V_{n+1}.

If $d_i = d$ for all i the corresponding rooted tree T_d is called d-regular.

Spherically homogeneous rooted tree



Automorphisms of $T_{\overline{d}}$

Let $\mathrm{Aut}(T_{\overline{d}})$ be the group of all automorphisms of $T_{\overline{d}}$ preserving the root. The boundary ∂T_d is the space of simple infinite paths in T_d starting at v_0. Let X_n be finite sets of cardinality d_n. Then $\partial T_{\overline{d}}$ is naturally isomorphic to $\prod_{n\in\mathbb{N}} X_n$. Equip $\partial T_{\overline{d}}$ with $\mathrm{Aut}(T_{\overline{d}})$-invariant Bernoulli measure $\mu = \mu_{\overline{d}}$.

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Any $g \in \mathrm{Aut}(T_{\overline{d}})$ can be written as

$$g = s(g_1, g_2, \ldots, g_{d_1}),$$

where $g_1, \ldots, g_{d_1}$ are the restrictions of g onto the subtrees emerging from the vertices of V_1 and s is a permutation on V_1.

Self-similar and automaton group

If $d_i = d$ for all i then elements $g_1, \dots, g_d$ can be viewed as elements of $\mathrm{Aut}(T_d)$.

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Definition

A subgroup $G < \mathrm{Aut}(T_d)$ is called self-similar if for any $g \in G$ one has $g_1, \ldots, g_d \in G$. A subgroup $G < \mathrm{Aut}(T_d)$ is called an automaton group if G is generated by a finite set S such that for any $g \in S$ one has $g_1, \ldots, g_d \in S$.

Examples: Grigorchuk group Γ

Grigorchuk group is the group $\Gamma =< a, b, c, d >$ acting on T_2 with

$$a = \epsilon, \ b = (a, c), \ c = (a, d), \ d = (1, b),$$

where ϵ is a non-trivial transformation of V_1. Γ is torsion free group of intermediate growth, subexponentally amenable bun not elementary amenable group.

Spectra of Γ

To study the spectrum of $\kappa(M)$ where $M = \frac{1}{4}(a+b+c+d)$ Bartholdi and Grigorchuk used operator recursions and associated the map

$$F(x,y) = \left(x - \frac{xy^2}{x^2-4}, \frac{2y^2}{x^2-4}\right)$$

to $\kappa(M)$. Studying F they obtained:

Theorem (Bartholdi-Grigorchuk)

$\sigma(\kappa(M)) = \sigma(\rho_x(M)) = [-\frac{1}{2}, 0] \cup [\frac{1}{2}, 1]$.

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Proposition (D.)

$\sigma(\rho_\Gamma(M)) = [-\frac{1}{2}, 0] \cup [\frac{1}{2}, 1]$.

Examples: Basilica group $\mathcal{B}$

Basilica group is the group $\mathcal{B} =< a, b >$ acting on T_2 with

$$a = (1, b),\ b = \epsilon(1, a).$$

Basilica group is the iterated monodromy group of $p(z) = z^2 - 1$, torsion free group of exponential grouth, amenable but not subexponentially amenable. Operator recursions for $\kappa(M)$ where $M = \frac{1}{4}(a + a^{-1} + b + b^{-1})$ give rise to the map

$$F(x, y) = \Big(\frac{y-2}{x^2}, -2 + \frac{y(y-2)}{x^2}\Big).$$

Examples: Lamplighter group

Grigorchuk and Zuk showed that the Lamplighter group $L = \mathbb{Z} \ltimes \mathbb{Z}_2^{\mathbb{Z}}$ can be realized as a group acting on a binary rooted tree as follows: $L =< a, b >$ where

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Theorem (Grigorchuk-Linnel-Schick-Zuk)

L is a counterexample to Strong Atiyah Conjecture.

Spectra of operators associated to actions on rooted trees

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Theorem (Bartholdi-Grigorchuk)

Let $G = < S >$ be a finitely generated group acting spherically transitively on a d-regular rooted tree T_d. Then for all $x \in \partial T_d$ one has $\rho_x \prec \kappa$ i.e. for every $m \in \mathbb{C}[G]$ one has

$$\sigma(\rho_x(m)) \subset \sigma(\kappa(m)).$$

If moreover the action of G on Gx is amenable then $\rho_x \sim \kappa$.

General case

Theorem (D.-Grigorchuk)

For an ergodic measure class preserving action of a countable group G on a standard Borel space (X, μ) one has:

$$\rho_x \sim \pi \prec \kappa$$

for almost all $x \in X$. If moreover (G, X, μ) is hyperfinite then $\pi \sim \kappa$.

Related results

Our results imply

Theorem (Kuhn)

For an ergodic Zimmer amenable measure class preserving action of G on a probability measure space (X, μ) one has

$$\kappa \prec \lambda_G,$$

where κ is the Koopman representation associated to the action of G and ρ_G is the regular representation.

Related results

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and the "only if" part of

Theorem (Pichot)

A measure class preserving action of a countable group G on a standard probability space $L^2(X, \mu)$ is hyperfinite if and only if for every $m \in l^1(G)$ with $\|m\|_1 = 1$ one has $\|\pi(m)\| = 1$, where π is the corresponding groupoid representation.

Spectral measures

Theorem (Spectral Theorem)

Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Then there exists a projector-valued measure $E(\lambda)$ supported on $\sigma(A)$ such that

$$A = \int \lambda \mathrm{d}E(\lambda).$$

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Given the projector-valued spectral measure $E(\lambda)$ and a vector $\xi \in \mathcal{H}$ the spectral measure η_ξ of A corresponding to ξ is

$$\eta_\xi(\lambda) = (E(\lambda)\xi, \xi).$$

Kesten spectral measures

Let G be finitely generated group with a symmetric generating set S and $M = \sum_{s \in S} s \in \mathbb{C}[G]$.

Definition

Given an action of G on a set X the spectral measure λ_x of the operator $\rho_x(M)$ corresponding to the vector $\delta_x \in l^2(Gx)$ is called Kesten spectral measure.

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Proposition (Kesten)

Let P_n be a decreasing sequence of finite index subgroups of G and $P = \cap P_n$. Then $\lambda_{G/P_n} \to \lambda_{G/P}$ weakly when $n \to \infty$.

Counting measures

For a bounded linear operator T and a subset $A \subset \mathbb{C}$ denote by $N(T, A)$ the number of eigenvalues of T inside A (counting multiplicity).

Definition

For a subgroup $H < G$ of finite index and a self-adjoint $m \in \mathbb{C}[G]$ introduce counting measures $\tau^m_{G/H}$ of $\rho_{G/H}(m)$:

$$\tau^m_{G/H}(A) = \frac{1}{|G/H|} N(\rho_{G/H}(m), A).$$

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If G is finitely generated with a symmetric set of generators S and $M = \frac{1}{|S|} \sum_{s \in S} s$ we set $\tau_{G/H} = \tau^M_{G/H}$.

Kesten-Neumann-Serre measures

Proposition (Bartholdi-Grigorchuk)

Let P_n be a decreasing sequence of finite index subgroups of a finitely generated group G. Then there exists a weak limit

$$\lim \tau_{G/P_n} = \tau_*.$$

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The measure τ_* is called Kesten-Neumann-Serre (KNS) measure.

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Let P_n be a decreasing sequence of finite index subgroups of a group G and $m \in \mathbb{C}[G]$ be self-adjoint. Then there exists a weak limit

$$\lim \tau^m_{G/P_n} = \tau^m_*.$$

.

Kesten and KNS measures for groups acting on rooted trees

Let $G < \mathrm{Aut}(T_d)$. For $x \in \partial T_d$ let $v_n = v_n(x) \in V_n$ be the sequence of vertices through which the path defined by x passes. Set $P_n = P_n(x) = \mathrm{St}_G(v_n)$.

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Remark

1) Consider the action of G on V_n. The quasi-regular representation ρ_{G/P_n} is isomorphic to ρ_{v_n}.
2) The counting measures τ^*_{G/P_n} and the corresponding KNS measure τ^m_* do not depend on $x \in \partial T_d$.

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Bartholdi and Grigorchuk caclulated KNS measures for Γ and some other automaton groups. They proposed a question under which conditions τ_* and λ_x coincide.

Kesten and KNS measures for groups acting on rooted trees

Proposition (Bartholdi-Grigorchuk)

For the action of Γ *on* T_2 *for almost all* $x \in T_2$ *the KNS measure* τ_* *coincides with the Kesten measure* λ_x.

Kesten and KNS measures for groups acting on rooted trees

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For the action of Γ on T_2 for almost all $x \in T_2$ the KNS measure τ_* coincides with the Kesten measure λ_x.

Theorem (Grigorchuk-Zuk, Kambites-Silva-Steinberg)

Let G be an automaton group with a symmetric set of generators S acting on T_d spherically transitively and essentially freely. Then the KNS measure τ_* coincides with the Kesten measures of ρ_G.

Action of arbitrary group on a rooted tree

Proposition (Grigorchuk)

Every finitely generated residually finite group has a faithful spherically transitive action on a spherically homogeneous rooted tree.

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Let G be any group and $\mathcal{P} = \{P_n\}_{n\in\mathbb{Z}_+}$ be a decreasing sequence of finite index subgroups of G with $P_0 = G$. Set $d_n = [P_{n-1} : P_n]$. Introduce a spherically homogeneous rooted tree $T_{\overline{d}}$:

- $V_n = G/P_n$;
- gP_n is connected to fP_{n+1} iff $gP_n \supset fP_{n+1}$.

G acts on each level by left multiplication: $g(fP_n) = gfP_n$. Let $\mu_{\mathcal{P}}$ be the $\mathrm{Aut}(T_{\overline{d}})$-invariant measure on $X_{\mathcal{P}} = \partial T_{\overline{d}}$

Relation between the measures

Given measure class preserving action of G on a standard Borel space (X, μ) consider the corresponding groupoid representation $\pi : G \to U(L^2(\mathcal{R}, \nu))$. Let $\xi = \delta_{x,y} \in L^2(\mathcal{R}, \nu)$. Fix $m \in \mathbb{C}[G]$. Introduce the spectral measure γ_π^m of $\pi(m)$ associated to ξ.

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$$\gamma_\pi^m = \int \lambda_x^m \mathrm{d}\mu(x).$$

Relation between the measures

Given measure class preserving action of G on a standard Borel space (X, μ) consider the corresponding groupoid representation $\pi : G \to U(L^2(\mathcal{R}, \nu))$. Let $\xi = \delta_{x,y} \in L^2(\mathcal{R}, \nu)$. Fix $m \in \mathbb{C}[G]$. Introduce the spectral measure γ_π^m of $\pi(m)$ associated to ξ.One has:

$$\gamma_\pi^m = \int \lambda_x^m \mathrm{d}\mu(x).$$

Theorem (D.)

Let G be a countable group, $m \in \mathbb{C}[G]$ be self-adjoint and $\mathcal{P} = \{P_n\}_{n \in \mathbb{Z}_+}$ be a decreasing sequence of finite index subgroups of G. Let π be the groupoid representation of G corresponding to the action on $(X_\mathcal{P}, \mu_\mathcal{P})$. The KNS measure τ_*^m coincides with the spectral measure γ_π^m.

Thank you!