Large and symmetric

Anton A. Klyachko

Vague prehistory

A theorem

Suppose that G is a group and H is its subgroup of finite index. Then. . .

A theorem

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Suppose that G is a group and H is its subgroup of finite index. Then...

Textbooks in group theory contain some simple facts allowing us to find a finite-index subgroup in G which is similar to, but better than H. In particular,

Vague prehistory

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H contains a *normal* finite-index subgroup of G

Vague prehistory

A theorem

Suppose that G is a group and H is its subgroup of finite index. Then...

H contains a *normal* finite-index subgroup of G (whose index divides $|G:H|!$)

.

Vague prehistory

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Vague prehistory

A theorem

Suppose that G is a group and H is its subgroup of finite index. Then...

H contains a *normal* finite-index subgroup of G .

if G is finitely generated, then H contains a *characteristic* finite-index subgroup.

Vague prehistory

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Suppose that G is a group and H is its subgroup of finite index. Then. . .

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if G is finitely generated, then H contains a *fully characteristic* finite-index subgroup.

Vague prehistory

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Suppose that G is a group and H is its subgroup of finite index. Then...

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if G is finitely generated, then H contains a *verbal* finite-index subgroup.

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H contains a *normal* finite-index subgroup of G .

if G is finitely generated, then H contains a *characteristic* finite-index subgroup.

if H is abelian, then G has a *characteristic* abelian finite-index subgroup.

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Suppose that G is a group and H is its subgroup of finite index. Then...

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if H is abelian, then G has a *characteristic* abelian finite-index subgroup.

The last fact is tricky...

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Suppose that G is a group and H is its subgroup of finite index. Then...

H contains a *normal* finite-index subgroup of G .

if G is finitely generated, then H contains a *characteristic* finite-index subgroup.

if H is abelian, then G has a *characteristic* abelian finite-index subgroup.

The last fact is tricky... The characteristic subgroup does not necessarily lie inside H.

Vague history. Some proofs

The last fact is tricky. . .

Theorem

if a group G has an abelian finite-index subgroup H, then G has a *characteristic* abelian finite-index subgroup N.

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Proof.

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if a group G has an abelian finite-index subgroup H, then G has a *characteristic* abelian finite-index subgroup N.

Proof.

Let us try to take $N_1 = \langle \varphi(H) \mid \varphi \in \mathrm{Aut}\, G \rangle$.

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Proof.

Let us try to take $N_1 = \langle \varphi(H) \mid \varphi \in \mathrm{Aut}\, G \rangle$. Well, N_1 is of finite index, characteristic but not abelian.

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if a group G has an abelian finite-index subgroup H, then G has a *characteristic* abelian finite-index subgroup N.

Proof.

Let us try to take $N_1 = \langle \varphi(H) \mid \varphi \in \operatorname{Aut} G \rangle$. Well, N_1 is of finite index, characteristic but not abelian. Then, let us take

$N_2 = Z(N_1)$.

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Let us try to take $N_1 = \langle \varphi(H) \mid \varphi \in \operatorname{Aut} G \rangle$. Well, N_1 is of finite index, characteristic but not abelian. Then, let us take

$N_2 = Z(N_1)$. It is abelian. . .

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index? Yes! Actually, $N_1 = \langle \varphi_1(H), \dots, \varphi_k(H) \rangle$ (because

$|G : H| < \infty$).

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$|G : H| < \infty$). So, $N_2 = Z(N_1) \supseteq \varphi_1(H) \cap \dots \cap \varphi_k(H)$. And the

index is finite.

Khukhro–Makarenko theorem (2007)

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An *outer* (or *multilinear*) commutator identity is an identity of the form $[\dots[x_1,\dots,x_t]\dots]=1$ with some meaningful arrangement of brackets, where all letters $x_1,\dots,x_t$ are different:

$$[[x,y],z]=1;\ [[x,y],[z,t]]=1;\ [[[x,y],[z,t]],u]=1;\dots$$

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Theorem (K, Melnikova, 2009)

There exist a one-page proof of the Khukhro–Makarenko theorem.

Generalisation

Khukhro–Makarenko theorem for algebras (2008)

Let G be an algebra (possibly, non-associative) over a field. If G contains a finite-codimensional subspace satisfying a multilinear identity, then G contains a finite-codimensional subspace satisfying the same identity and invariant under all automorphisms of G.

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Here, "*multilinear*" is in the usual sense:

$$(xy)t + 2y(xt) + 3t(yx) = 0, \dots$$

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. . . (Some theorem on multioperator groups that includes all these variants of the Khukhro–Makarenko theorem).

My favorite application

Very easy folklore theorem

If $N \triangleleft G$, N is solvable of derived length n and G/N is solvable of derived length m, then G is solvable of derived length $n+m$.

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What about virtually solvable?

(Khukhro, K, Makarenko, Melnikova, 2009)

If $N \triangleleft G$, N is virtually solvable of derived length n and G/N is virtually solvable of derived length m , then G is virtually solvable of derived length $n+m+1$.

Similar but...

Makarenko and Shumyatsky, 2012

Suppose that a locally finite group G contains a finite-index subgroup N having normal (in G) series

$$\{1\} = A_0 \subseteq \cdots \subseteq A_n = N$$

such that each quotient A_i/A_{i-1} either satisfies a multilinear commutator identity $w_i = 1$ or is locally nilpotent. Then G contains a characteristic subgroup H with the same property.

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. . . (Some theorem on multioperator groups that includes all known variants of the Khukhro–Makarenko theorem).

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The most general theorem

Theorem (K, Milentyeva, 2015)

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Applications. Groups, Series theorem.

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Suppose that a locally finite group contains a finite-index subgroup N having normal (in G) series $\{1\} = A_0 \subseteq \cdots \subseteq A_n = N$ such that each quotient A_i/A_{i-1} either satisfies a multilinear commutator identity $w_i = 1$ or is locally nilpotent. Then G contains a characteristic subgroup with the same property.

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K, Milentyeva, 2015

Here, locally finite may be replaced by *any*

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Here, locally finite may be replaced by *any* and

locally nilpotent may be replaced by *finite, finite p-group, locally finite, periodic, Noetherian, Artinian, nilpotent, solvable, virtually solvable, locally polycyclic, group satisfying nontrivial identities, group without nonabelian free subgroups, amenable,. . .* .

Locally nilpotent, finite, finite p-group, locally finite, periodic, Noetherian, Artinian, nilpotent, solvable, virtually solvable, locally polycyclic, group satisfying nontrivial identities, group without nonabelian free subgroups, amenable,. . .

Locally nilpotent, finite, finite p-group, locally finite, periodic, Noetherian, Artinian, nilpotent, solvable, virtually solvable, locally polycyclic, group satisfying nontrivial identities, group without nonabelian free subgroups, amenable,. . .

What is this list?

Locally nilpotent, finite, finite p-group, locally finite, periodic, Noetherian, Artinian, nilpotent, solvable, virtually solvable, locally polycyclic, group satisfying nontrivial identities, group without nonabelian free subgroups, amenable,...

What is this list?

This is just a (non-complete) list of *radical formations*, i.e. classes of groups closed with respect to normal subgroups, finite products of normal subgroups, homomorphic images, and subdirect products.

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In our dual theorem finite-index is replaced by *finite* and subgroup satisfying an outer commutator identity is replaced by *quotient group satisfying a universal positive first-order formula*,

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In our dual theorem finite-index is replaced by *finite* and subgroup satisfying an outer commutator identity is replaced by *quotient group satisfying a universal positive first-order formula*, e.g.,

$$(\forall x)(\forall y)\Big(\big(x^3 = y^3 \ \wedge\ (xy)^4 = (yx)^4\big) \ \vee\ (xy)^{2015} = 1 \ \vee\ [x, y]^5 = 1\Big).$$

Applications. Groups. “Co-Khukhro–Makarenko theorem”

Khukhro–Makarenko theorem (2007)

If a group has a finite-index subgroup satisfying an outer commutator identity, then this group also has a characteristic finite-index subgroup satisfying the same identity.

Dual theorem (K, Milentyeva, 2015)

If a group G has a finite normal subgroup such that, in the quotient group, a given universal positive closed first-order formula holds, then G has a characteristic finite subgroup with the same property.

Ring-theoretic analogue

(K, Milentyeva, 2015)

If an algebra G has a finite-dimensional two-sided ideal such that the quotient algebra satisfies a given universal positive closed first-order formula (in the language of algebras over the given field), then G has a characteristic finite-dimensional two-sided ideal with the same property.

Applications. Graphs

(K, Milentyeva, 2015)

Let $\{\Gamma_1, \ldots, \Gamma_l\}$ be a finite set of finite graphs called *forbidden* and considered up to isomorphism,

(K, Milentyeva, 2015)

Let $\{\Gamma_1, \ldots, \Gamma_l\}$ be a finite set of finite graphs called *forbidden* and considered up to isomorphism, and let G be some graph. If G

contains a finite set N of edges such that $G \setminus N$ does not contain forbidden subgraphs, then G contains a finite set of edges H which is invariant with respect to all automorphisms of G and has the same property: $G \setminus H$ does not contain forbidden subgraphs.

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Planarity theorem (K, Milentyeva, 2015)

If a graph can be made planar by removing a finite number of edges, then it can be made planar by removing a finite set of edges which is invariant with respect to all automorphisms of the graph.

Khukhro–Makarenko theorem (2007)

If a group has a finite-index subgroup N satisfying an outer commutator identity, then this group also has a characteristic finite-index subgroup H satisfying the same identity.

Estimates

Khukhro–Makarenko theorem (2007)

If a group has a finite-index subgroup N satisfying an outer commutator identity, then this group also has a characteristic finite-index subgroup H satisfying the same identity.

(K, Melnikova, 2009)

In addition,

$$\log_2 |G:H| \leqslant f^{t-1}(\log_2 |G:N|)$$

if the subgroup N is normal and, therefore, $\log_2 |G:H| \leqslant f^{t-1}(\log_2 |G:N|!)$ in the general case, where $f^k(x)$ is the k-th iteration of the function $f(x) = x(x+1)$.

Similar estimates are valid in all other theorems except one.

Estimates. The only exception

Planarity theorem (K, Milentyeva, 2015)

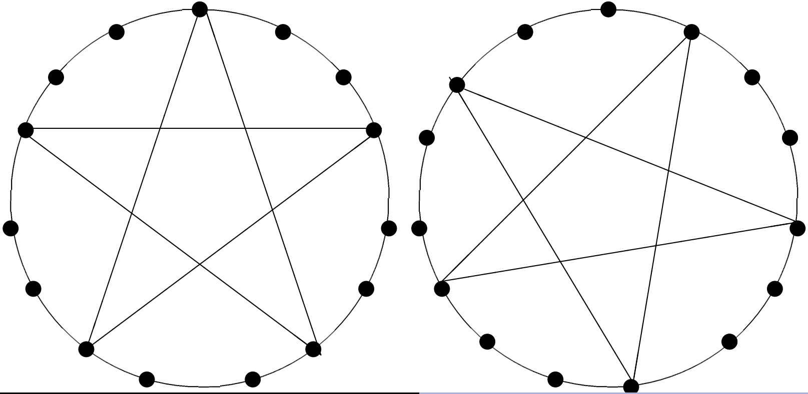
If a graph can be made planar by removing a finite set of edges N, then it can be made planar by removing a finite set of edges H which is invariant with respect to all automorphisms of the graph.

Estimates. The only exception

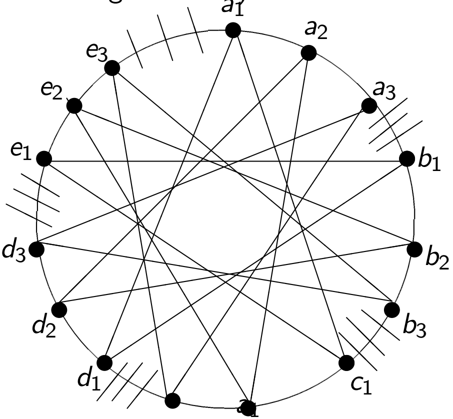
Planarity theorem (K, Milentyeva, 2015)

If a graph can be made planar by removing a finite set of edges N, then it can be made planar by removing a finite set of edges H which is invariant with respect to all automorphisms of the graph.

$|H|$ cannot be estimated by a function of $|N|$.



This is just the “union” of the 3 graphs above (where 3 plays the role of large number).



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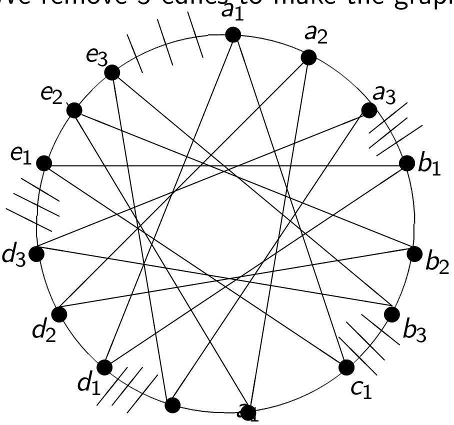
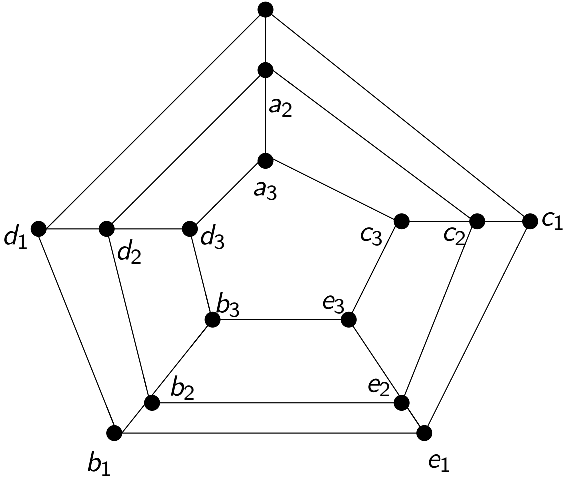
We remove 5 edhes to make the graph planar.

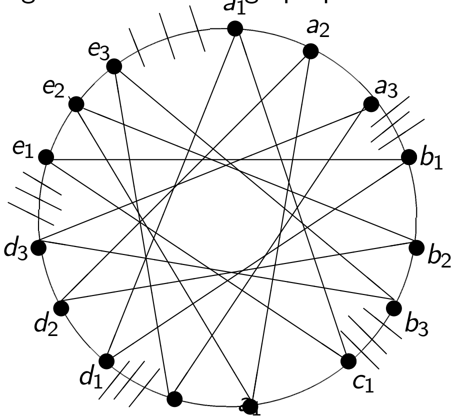


Picture

Yes. The obtained graph is planar; it is somorphic to

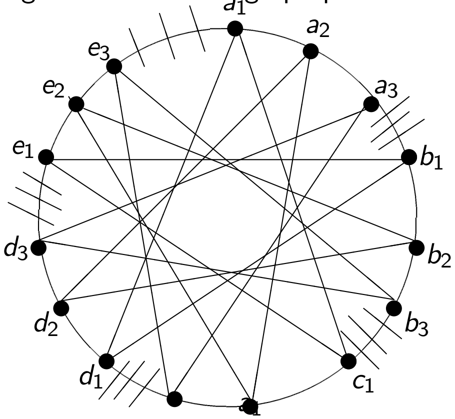


But we cannot remove an automorphism invariant small set of edges and make the graph planar.

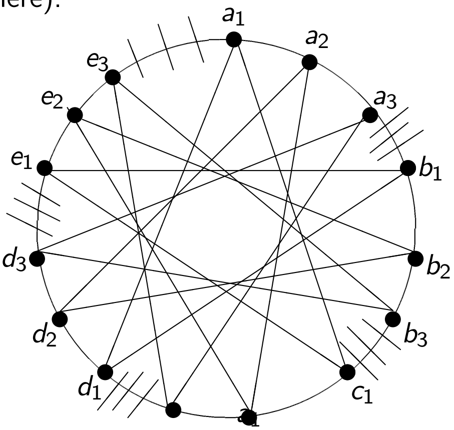


Picture

But we cannot remove an automorphism invariant small set of edges and make the graph planar.



But we cannot remove an automorphism invariant small set of edges and make the graph planar. Because the $Aut(G)$-orbit of each eges contains $\geqslant 3 \cdot 5$ edges (and 3 represents a large number here).



Bonus

Problem for high school

In the three-dimensional Euclidean space, there is a set X. It is known that we can remove a finite set of points from X in such a way that no 2015 of the remaining points lie on the same sphere. Show that this finite set can be chosen invariant under all symmetries($=$ isometries) of X.

Links and thanks

A lot of other applications can be found in our paper and the literature cited therein.

- Ant. A. Klyachko, Maria V. Milentyeva. Large and symmetric: The Khukhro-Makarenko theorem on laws – without laws. Journal of Algebra, 2015, 424, 222-241. See also arXiv:1309.0571.

Thank you!