Infinitely presented graphical small cancellation groups

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Motivation

Graphical small cancellation theory (Gromov 2003)

Tool for constructing finitely generated groups with prescribed subgraphs in their Cayley graphs.

Gromov's monsters (Gromov '03, Arzhantseva-Delzant '08, Osajda '14)

- Contain expanders (infinite sequences of sparse highly connected finite graphs) in a "good" way.
- Do not coarsely embed into Hilbert space.
- Only known counterexamples to the Baum-Connes conjecture with coefficients.

Today: most general combinatorial interpretation of the theory!

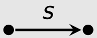
Plan

1. Graphical small cancellation theory
2. Coarse embedding theorem
3. Acylindrical hyperbolicity theorem

Group defined by a labelled graph

Labelled graph

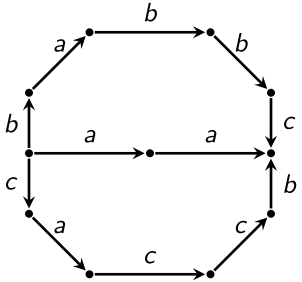
Graph Γ where every edge has orientation and label in S



Group defined by Γ

$$G(\Gamma) := \langle S \mid \text{labels of closed paths in } \Gamma \rangle$$

Example: $\langle a, b, c \mid a^2c^{-1}b^{-2}a^{-1}b^{-1}, a^2b^{-1}c^{-2}a^{-1}c^{-1}, \ldots \rangle$



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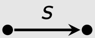
Graph Γ where every edge has orientation and label in S 

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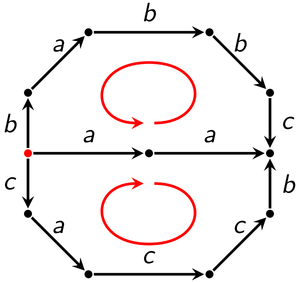

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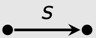
Graph Γ where every edge has orientation and label in S 

Group defined by Γ

$$G(\Gamma) := \langle S \mid \text{labels of closed paths in } \Gamma \rangle$$

Labelling induces map $\Gamma \to \mathrm{Cay}(G(\Gamma), S)$

Γ_0 connected component, choose image of a base vertex. Map:

$$\Gamma_0 \to \mathrm{Cay}(G(\Gamma), S).$$

Well-defined since closed paths are sent to closed paths.

Caution! Map is not necessarily injective, quasi-isometric, ...

Graphical $Gr(k)$ small cancellation condition

Piece

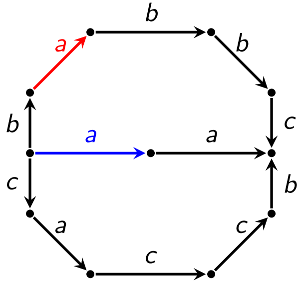
Reduced path p s.t. there exist paths p_1 and p_2 in Γ with the same label as p s.t. for every automorphism ϕ of Γ we have $\phi(p_1) \neq p_2$.

- pieces: $\xrightarrow{a}$

  $\xrightarrow{b}$

  $\xrightarrow{c}$

- $\mathrm{girth}(\Gamma) = 7$
- Γ satisfies $Gr(7)$-condition



Graphical $Gr(k)$-condition

- No simple closed path is concatenation of fewer than k pieces.
- The labelling is *reduced* (no: $\xrightarrow{s}\bullet\xleftarrow{s}$, $\xleftarrow{s}\bullet\xrightarrow{s}$).

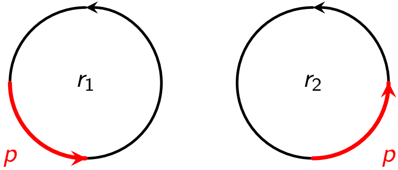
Example: classical $C(k)$-presentations

Classical small cancellation = graphical over cycle graphs

$\langle S \mid R \rangle$ presentation, for $r \in R$, let γ_r be cycle graph labelled by r.

$$\Gamma_R := \bigsqcup_{r \in R} \gamma_r$$

$\langle S \mid R \rangle$ classical $C(k)$. $\iff$ Γ_R graphical $Gr(k)$.



Classical pieces correspond to graphical pieces!

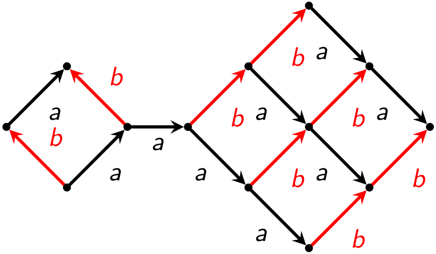
Method: van Kampen diagrams

Van Kampen diagram D over $\langle S \mid R \rangle$

Finite connected S-labelled graph embedded in $\mathbb{R}^2$ s.t. every bounded region (*face*) has boundary word in R. Boundary word of D is the word on the boundary of the unbounded region.

Van Kampen's Lemma

w is trivial in G. $\iff$ There exists D with boundary word w.



$\langle S \mid R \rangle = \langle a, b \mid aba^{-1}b^{-1} \rangle$

$w = b^{-1}a^4b^2a^{-2}b^{-2}a^{-1}ba^{-1}$

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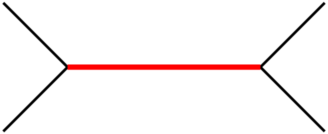
Arc in a diagram

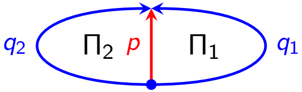
Path where all vertices except the endpoints have degree 2.



Small cancellation diagrams

Graphical van Kampen lemma (G 2012)

Γ a $Gr(6)$-graph, w trivial in $G(\Gamma)$. Then, if D is a diagram for w over $\langle S \mid$ labels of *simple* closed paths in $\Gamma\rangle$ with a minimal number of edges, then all interior arcs of D are pieces.



Maps

$$\begin{array}{ccccc} & & \partial\Pi_1 & \longrightarrow & \Gamma \\ & \nearrow & & & \\ p & & & & \\ & \searrow & & & \\ & & \partial\Pi_2 & \longrightarrow & \Gamma \end{array}$$

- If maps are distinct, then p is a piece.
- If maps coincide, then $q_1 q_2^{-1}$ is image of a closed path in Γ. Remove p and fold edges in diagram to decompose into images of simple closed paths.

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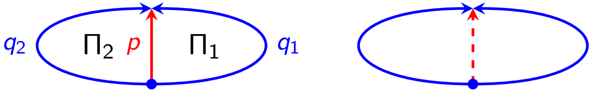

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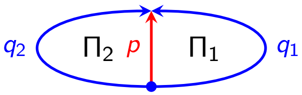

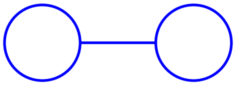

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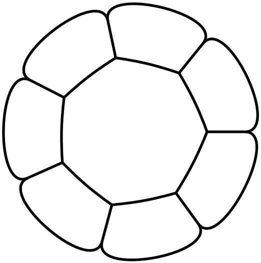
Γ a $Gr(k)$-graph for $k \geq 6$. Then every interior face of D has at least k arcs.



Application: Gromov hyperbolicity

Theorem (G 2012)

Γ a finite $Gr(7)$-labelled graph, S finite. Then $G(\Gamma)$ is hyperbolic.

Diagrams where all interior faces have ≥ 7 arcs satisfy

$$|\text{faces}(D)| \leq 8|\partial D|.$$

Thus, $G(\Gamma)$ has finite presentation satisfying linear isoperimetric inequality. $\square$

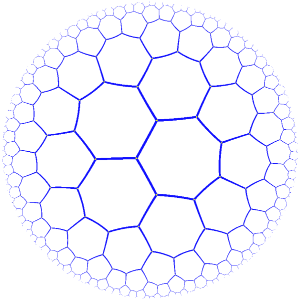


Image author: Tomruen (Wikipedia); colors have been altered

Application: asphericity

Theorem (G 2012)

Γ a $Gr(6)$-labelled graph with no non-trivial label-preserving automorphisms. Then $G(\Gamma)$ has an aspherical presentation complex and, hence, is torsion-free.

Idea: Diagrams where all faces have ≥ 6 arcs cannot tessellate the 2-sphere. $\square$

Remark. If $\Gamma = \Gamma_R$ with $\langle S \mid R \rangle$ classical $C(6)$, then no automorphisms in $\Gamma_R \leftrightarrow$ no proper powers in R.

Finitely presented graphical small cancellation groups

New torsion-free hyperbolic groups

- Property (T) groups (Gromov '03, Silberman '03, Ollivier-Wise '07).
- Non-unique product groups (Rips-Segev '87, Steenbock '15, G-Martin-Steenbock '15).

Plan

1 Graphical small cancellation theory

2 Coarse embedding theorem

3 Acylindrical hyperbolicity theorem

Coarse embedding theorem

Theorem (G 2012)

$\Gamma = \sqcup_{n\in\mathbb{N}}\Gamma_n$ a $Gr(6)$-graph, S finite, each Γ_n finite. Then Γ coarsely embeds into $\mathrm{Cay}(G(\Gamma), S)$.

Coarse embedding

A map $f : \sqcup_{n\in\mathbb{N}} X_n \to Y$, where X_n, Y are metric spaces, is a CE if for all sequences $(x_k, x'_k) \in \sqcup_{n\in\mathbb{N}} X_n \times X_n$ we have

$$d(x_k, x'_k) \to \infty \iff d\big(f(x_k), f(x'_k)\big) \to \infty$$

Applications

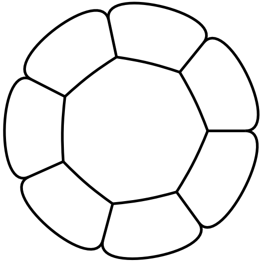
- F.g. group with CE expander is a counterexample to Baum-Connes conjecture with coefficients, does not CE into $\mathcal{H}$.
- F.g. group with CE large girth sequence of 3-regular graphs is not coarsely amenable.

Greendlinger's lemma

Lemma

D a diagram with at least 2 faces s.t. every interior face has at least 6 arcs. Then D has at least 2 faces that each intersect ∂D in a (connected!) arc and each have at most 3 interior arcs.

Proof: rewrite Euler characteristic $V - E + F = 1$.

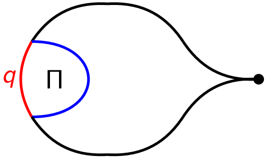


Step 1: each component injects

Lemma (G 2012)

Let Γ_0 be a connected component of $Gr(6)$-graph Γ. Then any label-preserving map $f : \Gamma_0 \to \mathrm{Cay}(G(\Gamma), S)$ is injective.

Assume $x \neq y$ with $f(x) = f(y)$. Let p path $x \to y$ s.t. $\exists$ diag. D for $\ell(p)$ whose number of edges is minimal among all choices for p.



Π face with arc $q = \Pi \cap \partial D$ subpath of ∂D. Maps $q \to p \to \Gamma$ and $q \to \partial\Pi \to \Gamma$. If maps are distinct, q is a piece. If maps coincide, remove q and replace it by q' in $D \to$ another path $p' : x \to y$, contradicting minimality. $\Rightarrow$ q piece & Π more than 3 interior arcs.

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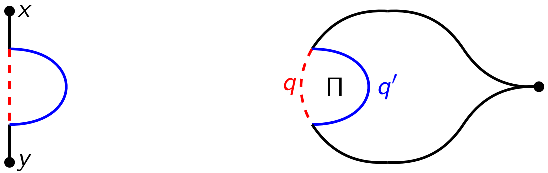

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Assume $x \neq y$ with $f(x) = f(y)$. Let p path $x \to y$ s.t. $\exists$ diag. D for $\ell(p)$ whose number of edges is minimal among all choices for p.

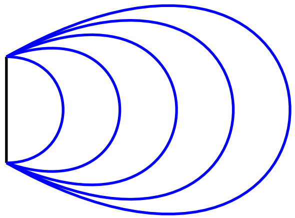
- D has at least 1 face because the labelling of Γ is reduced.
- Every face with $\Pi \cap \partial D$ connected has more than 3 interior arcs. (In particular, D has more than 1 face.)
- Contradiction to Greendlinger's lemma $\Rightarrow$ x and y do not exist $\Rightarrow$ f is injective.

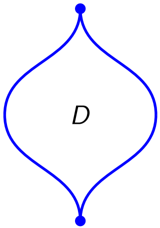
Step 2: prove coarse embedding

Theorem (G 2012)

$\Gamma = \sqcup_{n \in \mathbb{N}} \Gamma_n$ a $Gr(6)$-graph, S finite, each Γ_n finite. Then Γ coarsely embeds into $\mathrm{Cay}(G(\Gamma), S)$.

Embedded does not necessarily imply coarsely embedded:





Arguments as before show that diagrams D have no faces. □

$Gr(k)$-labellings exist

Theorem (Osajda 2014)

Let $k \geq 0$ and $(\Gamma_i)_{i\in\mathbb{N}}$ be a sequence of finite connected graphs with vertex degree $\leq d$ such that

- $|\Gamma_i| \to \infty$,
- $\operatorname{diam}(\Gamma_i)/\operatorname{girth}(\Gamma_i) < C$ for some $C < \infty$

Then there exist a finite set S and an infinite subsequence $(\Gamma_{j_i})_{i\in\mathbb{N}}$ such that $\sqcup_{i\in I}\Gamma_{j_i}$ admits a $Gr(k)$-labelling by S.

This produces

- Only known groups with coarsely embedded expanders, some even isometrically embedded (Ollivier '06).
- Only known non-coarsely amenable groups with the Haagerup property (Arzhantseva-Osajda '14).

Coarse embedding theorem

Theorem (G 2012)

$\Gamma = \sqcup_{n\in\mathbb{N}}\Gamma_n$ a $Gr(6)$-graph, S finite, each Γ_n finite. Then Γ coarsely embeds into $\mathrm{Cay}(G(\Gamma), S)$.

Research directions.

- $Gr(6)$-condition weakest possible condition to get coarse embedding $\rightsquigarrow$ explicit Gromov's monsters?
- New approach: use graphs with many automorphisms, e.g. sequences of finite Cayley graphs.

Plan

Acylindrical hyperbolicity theorem

Theorem (G-Sisto 2014)

Let Γ be a $Gr(7)$-labelled graph whose components are finite. Then $G(\Gamma)$ is either acylindrically hyperbolic or virtually cyclic.

Some consequences of acylindrical hyperbolicity

- G is SQ-universal.
- All asymptotic cones of G have cut-points.
- $C^*_{\text{red}}(G)$ is simple if G has no finite normal subgroups.

Acylindrical hyperbolicity

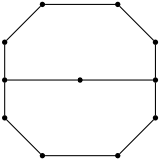
G is acylindrically hyperbolic if it is not virtually cyclic and acts by isometries on a Gromov hyperbolic space s.t. there exists a WPD element $g \in G$.

The hyperbolic space Y

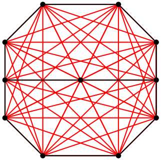
Theorem (GS 2014)

Γ a $Gr(7)$-labelled graph, W = all words read on Γ. Then $Y := \mathrm{Cay}(G(\Gamma), S \cup W)$ is Gromov hyperbolic.

Γ_0 in $\mathrm{Cay}(G(\Gamma), S)$:



Γ_0 in Y:



Proposition (GS 2014)

$G = \langle S | R \rangle$, $W = \{$subwords of elements of $R\}$. If $x \in F(S)$, let $|x|_{S \cup W}$ least k s.t. $x = w_1^{\pm 1} w_2^{\pm 1} \ldots w_k^{\pm 1}$, $w_i \in S \cup W$. If $\exists C > 0$:

$$\forall x \in F(S) \text{ representing } 1 \in G : \mathrm{Area}_R(x) < C|x|_{S \cup W},$$

then $\mathrm{Cay}(G, S \cup W)$ is hyperbolic.

The WPD element

WPD element

g is a WPD element for the action of G on Y if:

- g is hyperbolic, i.e. $\mathbb{Z} \to Y, z \mapsto g^z$ is a QI-embedding.
- g satisfies the WPD condition, i.e. for every $K > 0$ there exists $N_0 > 0$ such that for all $N \geq N_0$:

  $$\{h \in G \mid d_Y(h, 1) < K, d_Y(g^{-N} h g^N, 1) < K\} \text{ is finite.}$$

The WPD element: hyperbolicity

Sketch: definition of the WPD element g

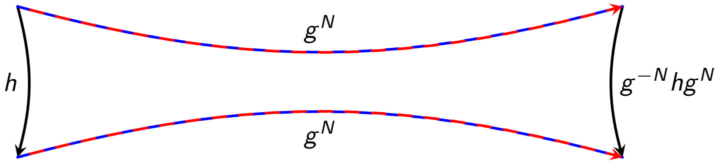
Γ_1 and Γ_2 distinct components of Γ, p_1 path in Γ_1 and p_2 path in Γ_2 such that both are not pieces. Define $g := \ell(p_1)\ell(p_2)$.

Sketch: hyperbolicity of g.

$\ell(p_1)\ell(p_2)$ cannot be read on any component of Γ. $\Rightarrow$ A path in $\mathrm{Cay}(G(\Gamma), S)$ labelled by $\big(\ell(p_1)\ell(p_2)\big)^N$ is not contained in $\ll N$ embedded components of Γ. Metric in Y counts how many embedded components of Γ one has to go through. $\Rightarrow$ $d(1, g^N)_Y \approx N$.

The WPD condition

To prove WPD condition, study quadrangular diagrams:



Acylindrical hyperbolicity theorem

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Research directions.

- Action of Gromov's monsters on cone-off space $\rightsquigarrow$ positive result about Baum-Connes?
- Study cone-off space for other limits of hyperbolic groups.
- Use (graphical) small cancellation groups to study class of acylindrically hyperbolic groups.

Conclusion

Graphical small cancellation theory

- General tool for constructing groups with prescribed (coarsely) embedded infinite subgraphs and, hence, extreme analytic properties.
- Lets us study these groups through actions on concrete hyperbolic spaces.
- Provides new examples for studying the class of acylindrically hyperbolic groups.

Further reading

D. Gruber, *Groups with graphical C(6) and C(7) small cancellation presentations*, Trans. Amer. Math. Soc. **367** (2015), no. 3, 2051–2078.

D. Gruber, *Infinitely presented* $C(6)$*-groups are SQ-universal*, J. London Math. Soc. **92** (2015), no. 1, 178–201.

D. Gruber, A. Martin, and M. Steenbock, *Finite index subgroups without unique product in graphical small cancellation groups*, Bull. London Math. Soc. **47** (2015), no. 4, 631–638.

D. Gruber and A. Sisto, *Infinitely presented graphical small cancellation groups are acylindrically hyperbolic*, arXiv:1408.4488 (2014).