Conjugacy growth and hyperbolicity

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Webinar, December 3, 2015

Summary

1. Conjugacy growth in groups
2. Conjugacy growth series in groups
3. Rivin's conjecture for hyperbolic groups
4. Conjugacy representatives in acylindrically hyperbolic groups

Counting conjugacy classes

Let G be a group with finite generating set X.

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Let G be a group with finite generating set X.

- Denote by $[g]$ the conjugacy class of $g \in G$ and by $|g|_c$ the conjugacy length of $[g]$, where $|g|_c$ is the length of the shortest $h \in [g]$, with respect to X.
- The conjugacy growth function is then

$$\sigma_{G,X}(n) := \sharp\{[g] \in G \mid |g|_c = n\}.$$

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- **Guba-Sapir (2010)**: asymptotics of the conjugacy growth function for $BS(1, n)$, the Heisenberg group on two generators, diagram groups, some HNN extensions.

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- **Conjecture (Guba-Sapir)**: most (excluding the Osin or Ivanov type 'monsters') groups of standard exponential growth should have exponential conjugacy growth.
- **Breuillard-Cornulier-Lubotzky-Meiri (2011)**: uniform exponential conjugacy growth for f.g. linear (non virt. nilpotent) groups.
- **Hull-Osin (2013)**: conjugacy growth not quasi-isometry invariant. Also, it is possible to construct groups with a prescribed conjugacy growth function.

Conjugacy growth in geometry

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A slight modification of the conjugacy growth function (including only the non-powers) appears in geometry:

- counting the primitive closed geodesics of bounded length on a compact manifold M of negative curvature and exponential volume growth gives, via quasi-isometries, good (exponential) asymptotics for $\sigma(n)$ for the fundamental group of M (Margulis, ...).

The conjugacy growth series

Let G be a group with finite generating set X.

- The conjugacy growth series of G with respect to X records the number of conjugacy classes of every length. It is

$$\widetilde{\sigma}_{(G,X)}(z) := \sum_{n=0}^{\infty} \sigma_{(G,X)}(n) z^n,$$

where $\sigma_{(G,X)}(n)$ is the number of conjugacy classes of length n.

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If G hyperbolic, then the conjugacy growth series of G is rational if and only if G is virtually cyclic.

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Theorem (Antolín-C., 2015)

If G is non-elementary hyperbolic, then the conjugacy growth series is transcendental.

$\Leftarrow$

Theorem (C., Hermiller, Holt, Rees, 2014)

Let G be a virtually cyclic group. Then the conjugacy growth series of G is rational.

NB: Both results hold for all symmetric generating sets of G.

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1. $\pi(w) \in [g]$, where $\pi \colon X^* \to G$ the natural projection, and
2. $l(w) = |\pi(w)| = |\pi(w)|_c$ is of minimal length in $[g]$, where
   - $l(w) :=$ word length of $w \in X^*$
   - $|g| = |g|_X :=$ the (group) length of $g \in G$ with respect to X.

Conjugacy growth series in virt. cyclic groups: $\mathbb{Z}$, $\mathbb{Z}_2 * \mathbb{Z}_2$

In $\mathbb{Z}$ the conjugacy growth series is the same as the standard one:

$$\widetilde{\sigma}_{(\mathbb{Z},\{1,-1\})}(z) = 1 + 2z + 2z^2 + \cdots = \frac{1+z}{1-z}.$$

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In $\mathbb{Z}_2 * \mathbb{Z}_2$ a set of conjugacy representatives is $1, a, b, ab, abab, \ldots$, so

$$\widetilde{\sigma}_{(\mathbb{Z}_2 * \mathbb{Z}_2,\{a,b\})}(z) = 1 + 2z + z^2 + z^4 + z^6 \cdots = \frac{1+2z-2z^3}{1-z^2}.$$

Conjugacy growth series in free groups: $F_2 = \langle a, b \rangle$

Set $a < b < a^{-1} < b^{-1}$ and choose as conjugacy representative the smallest shortlex rep. in each conjugacy class, so the language is

$$\{a^{\pm k}, b^{\pm k}, ab, ab^{-1}, ba^{-1}, a^{-1}b^{-1}, a^2b, \cancel{aba}, \cdots\}$$

Asymptotics of conjugacy growth in the free group

Idea: take all cyclically reduced words of length n, whose number

is $(2k-1)^n + 1 + (k-1)[1 + (-1)^n]$, and divide by n.

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Coornaert, 2005: For the free group F_k, the primitive (non-powers) conjugacy growth function is given by

$$\sigma_p(n) \sim \frac{(2k-1)^{n+1}}{2(k-1)n} = C\frac{e^{\mathbf{h}n}}{n},$$

where $C = \frac{2k-1}{2(k-1)}$, $h = \log(2k-1)$.

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In general, when powers are included, one cannot divide by n.

The conjugacy growth series in free groups

- Rivin (2000, 2010): the conjugacy growth series of F_k is not rational:

$$\widetilde{\sigma}(z) = \int_0^z \frac{\mathcal{H}(t)}{t} dt, \quad \text{where}$$

$$\mathcal{H}(x) = 1 + (k-1)\frac{x^2}{(1-x^2)^2} + \sum_{d=1}^{\infty} \phi(d) \left(\frac{1}{1-(2k-1)x^d} - 1 \right).$$

Free products of finite groups

Theorem (C. - Hermiller, 2012)

For A, B finite groups with generating sets $X_A = A \setminus 1_A$, $X_B = B \setminus 1_B$,

and $A * B$ with generating set $X = X_A \cup X_B$.

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and $A * B$ with generating set $X = X_A \cup X_B$.

Then $\widetilde{\sigma}(A * B, X)$ is rational iff $A = B = \mathbb{Z}/2\mathbb{Z}$, i.e. $A * B = D_\infty$.

Rational, algebraic, transcendental

A generating function $f(z)$ is

- rational if there exist polynomials $P(z)$, $Q(z)$ with integer coefficients such that $f(z) = \frac{P(z)}{Q(z)}$;
- algebraic if there exists a polynomial $P(x, y)$ with integer coefficients such that $P(z, f(z)) = 0$;
- transcendental otherwise.

Rivin's conjecture ⇒

If G is non-elementary hyperbolic, then the conjugacy growth series $\widetilde{\sigma}$ is not rational.

Rivin's conjecture $\Rightarrow$

If G is non-elementary hyperbolic, then the conjugacy growth series $\widetilde{\sigma}$ is not rational.

Proof. (Antolín-C., 2015)

- Recall: $\sigma(n) := \sharp\{[g] \in G \mid |g|_c = n\}$ is the strict conjugacy growth.
- Let $\phi(n) := \sharp\{[g] \in G \mid |g|_c \leq n\}$ be the cumulative conjugacy growth.

Theorem [AC] (Conjugacy bounds based on Coornaert and Knieper).

Let G be a non-elementary word hyperbolic group. Then there are positive constants A, B and n_0 such that

$$A\frac{e^{\mathbf{h}n}}{n} \leq \phi(n) \leq B\frac{e^{\mathbf{h}n}}{n}$$

for all $n \geq n_0$, where $\mathbf{h}$ is the growth rate of G, i.e. $e^{\mathbf{h}n} = |Ball(n)|$.

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MESSAGE:.

The number of conjugacy classes in the ball of radius n is asymptotically the number of elements in the ball of radius n divided by n.

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Lemma (Flajolet: Trancendence of series based on bounds).

Suppose there are positive constants $A, B, \mathbf{h}$ and an integer $n_0 \geq 0$ s.t.

$$A\frac{e^{\mathbf{h}n}}{n} \leq a_n \leq B\frac{e^{\mathbf{h}n}}{n}$$

for all $n \geq n_0$. Then the power series $\sum_{i=0}^{\infty} a_n z^n$ is not algebraic.

Bounds for the conjugacy growth

Let $\phi_p(n) := \sharp\{\text{primitive } [g] \in G \mid |g|_c \ \leq \ n\}$ be the primitive cumulative conjugacy growth.

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Let G be a non-elementary word hyperbolic. Then there are positive constants A and n_0 such that for all $n \geq n_0$

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Let $\phi_p(n) := \sharp\{\text{primitive } [g] \in G \mid |g|_c \leq n\}$ be the primitive cumulative conjugacy growth.

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Let G be a non-elementary word hyperbolic. Then there are positive constants A and n_0 such that for all $n \geq n_0$

$$A\frac{e^{\mathbf{h}n}}{n} \leq \phi_p(n).$$

Theorem.(Coornaert and Knieper, IJAC 2004)

Let G be a torsion-free non-elementary word hyperbolic group. Then there are positive constants B and n_1 such that for all $n \geq n_1$

$$\phi_p(n) \leq B\frac{e^{\mathbf{h}n}}{n}.$$

Rivin's conjecture $\Rightarrow$: Proof

1. Drop torsion requirement from upper bound of Coornaert and Knieper:
   (i) use the fact that there exists $m < \infty$ such that all finite subgroups $F \leq G$ satisfy $|F| \leq m$.
   (ii) most ($\geq \frac{n}{m}$) cyclic permutations of a primitive conjugacy representative of length n correspond to different elements of length n in G.

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2. Find conjugacy growth upper bound for all conjugacy classes, i.e. include the non-primitive classes in the count.

■

Next steps: generalize

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   (a) we need sharp bounds for the standard growth function [Yang] $\surd$

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   (a) we need sharp bounds for the standard growth function [Yang] $\surd$

   (b) we need sharp bounds for the conjugacy growth function.

2. Rivin's conjecture for acylindrically hyperbolic groups:

   Is the conjugacy growth series of a f.g. acylindrically hyperbolic group transcendental?

Conjugacy representatives in acylindrically hyperbolic groups

Formal languages and the Chomsky hierarchy

Let X be a finite alphabet. A formal language over X is a set $L \subset X^*$ of words.

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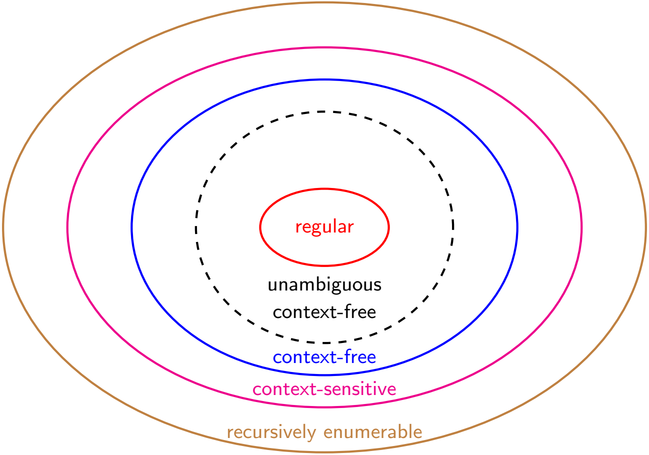

Formal languages and their algebraic complexity

Let $L \subset X^*$ be a language.

- The growth function $f_L : \mathbb{N} \to \mathbb{N}$ of L is:

$$f_L(n) = \sharp\{w \in L \mid w \text{ of length } n\}.$$

- The growth series of L is

$$\mathcal{S}_L(z) = \sum_{n=0}^{\infty} f_L(n) z^n.$$

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Theorem

- Regular languages have RATIONAL growth series.
- Unambiguous context-free languages have ALGEBRAIC growth series.

  (Chomsky-Schützenberger)

Consequences of the Rivin conjecture

Corollary. [AC]

Let G be a non-elementary hyperbolic group, X a finite generating set and $\mathcal{L}_c$ any set of minimal length representatives of conjugacy classes.

Then $\mathcal{L}_c$ is not regular.

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Let G be a non-elementary hyperbolic group, X a finite generating set and $\mathcal{L}_c$ any set of minimal length representatives of conjugacy classes.

Then $\mathcal{L}_c$ is not regular.

By Chomsky-Schüzenberger, $\mathcal{L}_c$ is not unambiguous context-free (UCF).

Acylindrically hyperbolic groups

Main Theorem [AC, 2015]

Let G be an acylindrically hyperbolic group, X any finite generating set, and $\mathcal{L}_c$ be a set containing one minimal length representative of each conjugacy class.

Then $\mathcal{L}_c$ is not unambiguous context-free, so not regular.

Acylindrically hyperbolic groups

Main Theorem [AC, 2015]

Let G be an acylindrically hyperbolic group, X any finite generating set, and $\mathcal{L}_c$ be a set containing one minimal length representative of each primitive conjugacy class/commensurating class.

Then $\mathcal{L}_c$ is not unambiguous context-free, so not regular.

Examples of acylindrically hyperbolic groups

(Dahmani, Guirardel, Osin, Hamenstädt, Bowditch, Fujiwara, Minasyan ...)

- relatively hyperbolic groups,
- all but finitely many mapping class groups of punctured closed surfaces,
- $\mathrm{Out}(F_n)$ for $n \geq 2$,
- directly indecomposable right-angled Artin groups,
- one-relator groups with at least 3 generators,
- most 3-manifold groups,
- lots of groups acting on trees,
- $C'(\frac{1}{6})$ small cancellation groups.

Acylindrically hyperbolic groups: definition 1

An action $\circ$ of a group G on a metric space $(\mathcal{S}, d)$ is called acylindrical if for every $\epsilon > 0$ there exist $R \geq 0$ and $N \geq 0$ such that for every two points $x, y \in \mathcal{S}$ with $d(x, y) \geq R$ there are at most N elements of G satisfying

$$d(x, g \circ x) \leq \epsilon \qquad \text{and} \qquad d(y, g \circ y) \leq \epsilon.$$

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A group G is called acylindrically hyperbolic if it admits a non-elementary acylindrical action on a hyperbolic space, where non-elementary is equivalent to G being non-virtually cyclic and the action having unbounded orbits.

Acylindrically hyperbolic groups: definition 2

A group is acylindrically hyperbolic if and only if it has a non-degenerate hyperbolically embedded subgroup in the sense of Dahmani, Guirardel and Osin.

Properties of a hyperbolically embedded subgroup:

- finitely generated,
- Morse (for any $\lambda \geq 1, c \geq 0$ there exists $\kappa = \kappa(\lambda, c)$ s. t. every (λ, c)-quasi-geodesic in $\Gamma(G, X)$ with end points in H lies in the κ-neighborhood of H),
- almost malnormal,
- quasi-isometrically embedded.

Main Theorem: Conjugacy representatives in an acylindrically hyperbolic group G are not regular (not UCF).

Idea of proof:

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(3) conjugators of conjugacy geodesics can be uniformly bounded*, and

(4) transform the language (1) for H into a language of conjugacy reps in G via regular operations using (3).

BCD: Bounded Conjugacy Diagrams

A group (G, X) satisfies K-(BCD) if there is a constant $K > 0$ such that for any pair of cyclic geodesic words U and V over X representing conjugate elements either

(a) $max\{|U|, |V|\} \leq K$,

or

(b) there is a word C over X, $|C| \leq K$, with $CU'C^{-1} =_G V'$, where U' and V' are cyclic shifts of U and V.

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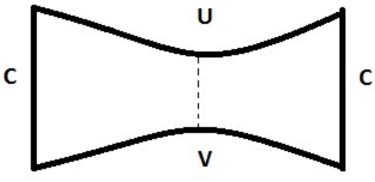
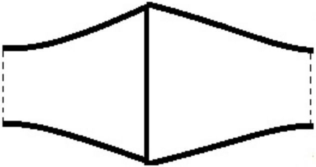
BCD appears in Bridson & Haefliger's book *Metric spaces of non-positive curvature*; they show that hyperbolic groups have BCD.

Short conjugator of U and V after cyclic permutations





Relative BCD

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We say that (G, X) has BCD relative to H if there is a $K \geq 0$ such that for any conjugacy geodesic U conjugate to an element in H we can find $g \in B_X(K)$ and a cyclic permutation U' of U so that $U' =_G g^{-1}Vg$, where $V \in H$.

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Suppose $\exists$ $K > 0$ such that G has K-BCD relative to H. $(**)$

Then any language of conjugacy representatives in G is not regular (UCF).

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- (DGO) a virtually free group H that is hyperbolically embedded in G, and
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Remark: In other words, acylindrically hyperbolic groups satisfy the conditions $(*)$ and $(**)$ in Result 1.

Result 1 (about languages)

$G = \langle X \rangle$

Suppose

$(*)$ $H \leqslant G$ is hyperbolic, qi embedded in G, almost malnormal and Morse.

$(**)$ G has BCD relative to H.

Then any language of conjugacy representatives in G is not regular (UCF).

Result 1: idea of proof

0. Remove all torsion conjugacy classes (finitely many) from the discussion.

0'. Today assume torsion-free G.

Sketch of proof - Step 1: strengthen the BCD condition

Construct a generating set Y for H s.t. to every conjugacy geodesic U over X, $U \in H^G$, we can associate a conj. geod. V over Y, where $V = g^{-1}Ug$ and

(a) the length of the conjugator g is uniformly bounded, and

(b) U and V 'fellow travel'.

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(a) the length of the conjugator g is uniformly bounded, and

(b) U and V 'fellow travel'.

Remarks:

(1) Call such a pair (U, V) a BCD pair.

(2) The fellow traveler property is non-standard, as U and V are words over different alphabets.

Step 1: The formal setup

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Let $B := (X \cup \$) \times (Y \cup \$)$ and suppose there are maps

$$X \mapsto G,\, Y \mapsto G \text{ with } \$ \mapsto 1_G.$$

Def. A pair $(U, V) \in B^*$ is a BCD pair with constant K if $V = g^{-1}Ug$,

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Step 1: The formal setup

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Sketch of proof

Lemma.

Let $K \geq 0$. The following set is a regular language:

$$\mathcal{M} = \{(U, V) \in B^* \mid \ (U, V) \text{ is a BCD pair with constant } K\}.$$

Step 1. Associate to each conjugacy geodesic U (over X) some V (over Y) such that (U, V) is a BCD pair.

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Standard Lemma.

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Lemma. The language

$$\mathcal{M}_2 = \{(U, V) \in B^* \mid V \equiv \min_{\leq_{\text{lex}}}(V' \mid (U, V') \text{ is a BCD pair})\}$$

is regular.

Define the map Δ by $\Delta(U) = V$, where V is such that $(U, V) \in \mathcal{M}_2$.

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Corollary. If $\mathcal{L}$ is regular (UCF) then $\mathcal{R}$ is regular (UCF).

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By construction $\mathcal{R}$ contains an H-representative of each G-conjugacy class. By malnormality $\mathcal{R}$ contains exactly one representative of each H-conjugacy class.

$\implies \mathcal{R}$ is a language of conjugacy representatives for the hyperbolic group H.

Conclusion

So if $\mathcal{L}$ ($=$ the conjugacy reps for G) were UCF, then $\mathcal{R}$ ($=$ the conjugacy reps. for H) would be UCF.

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So if $\mathcal{L}$ (= the conjugacy reps for G) were UCF, then $\mathcal{R}$ (= the conjugacy reps. for H) would be UCF.

This contradicts Rivin's conjecture, because H is hyperbolic. Thus conjugacy representatives in acylindrically hyperbolic groups cannot be unambiguous context-free. ■

Question: What type of language are they?

Thank you!

Rivin's conjecture $\Leftarrow$

Theorem (C., Hermiller, Holt, Rees, 2014)

Let G be a virtually cyclic group. Then for all generating sets of G the language of shortlex conjugacy representatives ConjSL is regular and hence the conjugacy growth series is rational.

Proof: We may assume that G is infinite.

- $\exists H \trianglelefteq G$, $H = \langle x \rangle \cong \mathbb{Z}$, with G/H finite.
- Let $C := C_G(H)$ be the centralizer of H in G.

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- Let $C := C_G(H)$ be the centralizer of H in G.
- The conjugation action of G on H defines a map $G \to \mathrm{Aut}(\mathbb{Z})$ with kernel C and so $|G : C| \leq 2$.
- For $g \in G \setminus C$, we have $gxg^{-1} = x^{-1} \Rightarrow x^{-1}gx = gx^2$, and hence the coset Hg is either a single conjugacy class in $\langle H, g \rangle$ (if $G \cong \mathbb{Z}$) or the union $[g] \cup [gx]$ (because $gx^k = x^{-1}(gx^{k-2})x$).

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- So $G \setminus C$ consists of finitely many conjugacy classes of G.
- Since $|\text{ConjSL} \cap (G \setminus C)| < \infty$, to prove regularity of ConjSL it is enough to show that $\text{ConjSL} \cap C$ is regular.

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- For $g \in C$, $|G : C_G(g)| < \infty$, so C is a union of infinitely many finite conjugacy classes.
- Let T be a transversal of H in G.
- Then for each $c \in C$, the conjugacy class of c is $\{t^{-1}ct \mid t \in T\}$, and hence any word w with $\pi(w) = c$ is in ConjSL $\Leftrightarrow$ there does not exist $t \in T$ for which $t^{-1}wt$ has a representative v with $v <_{sl} w$.

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- G hyperbolic $\implies$

$$L_1(t) := \{(u, v) : u, v \in \text{Geo}, \quad \pi(v) = \pi(t^{-1}ut)\}$$

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- So ConjSL $\cap$ C is the intersection of $\pi^{-1}(C)$ with

$$\text{Geo} \setminus \cup_{t \in T}(\{u \in \text{Geo} : \exists v \in \text{Geo such that } (u, v) \in L_1(t),\ v <_{sl} u\}).$$

- $|G : C|$ finite implies that $\pi^{-1}(C)$ is regular, so ConjSL $\cap$ C is also regular.

■