Branch groups: groups that look like trees

Alejandra Garrido

University of Oxford
garridoangul@maths.ox.ac.uk

Group Theory International Webinar, 4 December 2014
Outline

1. Introduction

2. Self-similarity

3. Branch structure
Trees

**Definition**

\((m_n)_{n \geq 0}\) sequence of integers \(\geq 2\).

\(T\) is a rooted tree of type \((m_n)_n\) if \(T\) is a tree with root \(v_0\) of degree \(m_0\) s.t. every vertex at distance \(n \geq 1\) from \(v_0\) has degree \(m_n + 1\).

\(V_n\) = vertices at distance \(n\) from root
\(T_v\) is subtree rooted at \(v\)
Groups that act on infinite rooted trees

Came to prominence from 1980s.
Groups that act on infinite rooted trees

Came to prominence from 1980s.

- Used as counterexamples/solutions to open problems in group theory

- General Burnside Problem (Aleshin, Grigorchuk, Gupta–Sidki)
- Groups of intermediate word growth (Grigorchuk)
- Non-uniform exponential word growth (Wilson)
- Amenable but not elementary amenable groups (Grigorchuk)
- Filling gaps in subgroup growth spectrum (Segal)

Links with dynamics and fractals (first sense in which they look like trees)

Regular trees are self-similar/fractal. Many of these groups are also “self-similar”. Self-similar groups (=groups generated by automata) appear naturally as iterated monodromy groups of self-coverings of topological spaces and encode combinatorial information about the dynamics of these coverings (Nekrashevych).
Groups that act on infinite rooted trees

Came to prominence from 1980s.

- Used as counterexamples/solutions to open problems in group theory
  - General Burnside Problem (Aleshin, Grigorchuk, Gupta–Sidki)
  - Groups of intermediate word growth (Grigorchuk)
  - Non-uniform exponential word growth (Wilson)
  - Amenable but not elementary amenable groups (Grigorchuk)
  - Filling gaps in subgroup growth spectrum (Segal)

Links with dynamics and fractals (first sense in which they look like trees)

Regular trees are self-similar/fractal. Many of these groups are also self-similar. Self-similar groups (=groups generated by automata) appear naturally as iterated monodromy groups of self-coverings of topological spaces and encode combinatorial information about the dynamics of these coverings (Nekrashevych).
Groups that act on infinite rooted trees

Came to prominence from 1980s.

- Used as counterexamples/solutions to open problems in group theory
  - General Burnside Problem (Aleshin, Grigorchuk, Gupta–Sidki)
  - Groups of intermediate word growth (Grigorchuk)
  - Non-uniform exponential word growth (Wilson)
  - Amenable but not elementary amenable groups (Grigorchuk)
  - Filling gaps in subgroup growth spectrum (Segal)

- Links with dynamics and fractals (first sense in which they look like trees)
Groups that act on infinite rooted trees

Came to prominence from 1980s.

- Used as counterexamples/solutions to open problems in group theory
  - General Burnside Problem (Aleshin, Grigorchuk, Gupta–Sidki)
  - Groups of intermediate word growth (Grigorchuk)
  - Non-uniform exponential word growth (Wilson)
  - Amenable but not elementary amenable groups (Grigorchuk)
  - Filling gaps in subgroup growth spectrum (Segal)
- Links with dynamics and fractals (first sense in which they look like trees)
  Regular trees are self-similar/fractal. Many of these groups are also “self-similar”. Self-similar groups (=groups generated by automata) appear naturally as iterated monodromy groups of self-coverings of topological spaces and encode combinatorial information about the dynamics of these coverings (Nekrashevych).
Example: Gupta–Sidki $p$-groups

$$T = T(p), \ p = \text{odd prime}$$

$$a := (12 \ldots p) \text{ on } V_1$$

$$b := (a, a^{-1}, 1, \ldots, 1, b).$$
Example: Gupta–Sidki $p$-groups

$T = T(p), \ p = \text{odd prime}$

\[ a := (12 \ldots p) \text{ on } V_1 \]
\[ b := (a, a^{-1}, 1, \ldots, 1, b). \]

$G := \langle a, b \rangle$
Example: Gupta–Sidki $p$-groups

$T = T(p), \; p = \text{odd prime}$

$a := (12 \ldots p) \; \text{on} \; V_1$

$b := (a, a^{-1}, 1, \ldots, 1, b)$.  

$G := \langle a, b \rangle$
1. Introduction

2. Self-similarity

3. Branch structure
For $G$ acting faithfully on $T$:

$\text{St}_G(v) := \{g \in G : vg = v\}$ is the stabilizer of $v$;

$\text{St}_G(n) := \bigcap_{v \in V_n} \text{St}_G(v)$ is the $n$th level stabilizer.
Definition

For $G$ acting faithfully on $T$:

$\text{St}_G(v) := \{ g \in G : vg = v \}$ is the stabilizer of $v$;

$\text{St}_G(n) := \bigcap_{v \in V_n} \text{St}_G(v)$ is the $n$th level stabilizer.

For any vertex $v$, for every $x \in \text{St}_G(v)$ we can assign a unique $x_v \in \text{Aut}(T_v)$ by restriction:

$$x_v := x|_{T_v}.$$
**Definition**

For $G$ acting faithfully on $T$:

$\text{St}_G(v) := \{ g \in G : vg = v \}$ is the stabilizer of $v$;

$\text{St}_G(n) := \bigcap_{v \in V_n} \text{St}_G(v)$ is the $n$th level stabilizer.

For any vertex $v$, for every $x \in \text{St}_G(v)$ we can assign a unique $x_v \in \text{Aut}(T_v)$ by restriction:

$$x_v := x|_{T_v}.$$ 

If $v \in V_n$, identify $T_v$ and $T(n)$ (tree rooted at level $n$).
For $G$ acting faithfully on $T$:

$\text{St}_G(v) := \{g \in G : vg = v\}$ is the stabilizer of $v$;

$\text{St}_G(n) := \bigcap_{v \in V_n} \text{St}_G(v)$ is the $n$th level stabilizer.

For any vertex $v$, for every $x \in \text{St}_G(v)$ we can assign a unique $x_v \in \text{Aut}(T_v)$ by restriction:

$$x_v := x|_{T_v}.$$ 

If $v \in V_n$, identify $T_v$ and $T(n)$ (tree rooted at level $n$). Then we have a homomorphism $\varphi_v : \text{St}(v) \to \text{Aut}(T(n))$, $x \mapsto x_v$. 
**Definition**

For $G$ acting faithfully on $T$:

\[
\text{St}_G(v) := \{g \in G : vg = v\}
\]
is the stabilizer of $v$;

\[
\text{St}_G(n) := \bigcap_{v \in V_n} \text{St}_G(v)
\]
is the $n$th level stabilizer.

For any vertex $v$, for every $x \in \text{St}_G(v)$ we can assign a unique $x_v \in \text{Aut}(T_v)$ by restriction:

\[
x_v := x|_{T_v}.
\]

If $v \in V_n$, identify $T_v$ and $T_{(n)}$ (tree rooted at level $n$). Then we have a homomorphism $\varphi_v : \text{St}(v) \to \text{Aut}(T_{(n)}), x \mapsto x_v$.

**Definition**

$G_v := \varphi_v(\text{St}_G(v))$ is the vertex section/projection of $G$ at $v$. 
We can think of $G_v$ as a subgroup of $\text{Aut}(T)$ (after identifying $T_v$ and $T$), but $G_v$ is not necessarily a subgroup of $G$. 

We say that $G$ is self-similar if $G_v \leq G$ for every $v \in T$. It is fractal/self-replicating if $G_v = G$ for every $v \in T$.

Example. Gupta–Sidki $p$-group is fractal: $\text{St}(1) = \langle b, b^a, \ldots, b^{a^{p-1}} \rangle$ where $b = (a, a^{-1}, 1, \ldots, 1), b^a = (b, a, a^{-1}, 1, \ldots, 1), \ldots, b^{a^{p-1}} = (a^{-1}, 1, \ldots, 1, b, a)$.

Look at $v =$ left-most vertex in first level; then $\phi_v(b) = a, \phi_v(b^a) = b$, so $G_v = G$. Similarly for rest of first level. So we have $\text{St}(1)$ subdirect in $G \times p$.
We can think of $G_v$ as a subgroup of $\text{Aut}(T)$ (after identifying $T_v$ and $T$), but $G_v$ is not necessarily a subgroup of $G$.

We say that $G$ is self-similar if $G_v \leq G$ for every $v \in T$. It is fractal/self-replicating if $G_v = G$ for every $v \in T$. 
We can think of $G_v$ as a subgroup of Aut($T$) (after identifying $T_v$ and $T$), but $G_v$ is not necessarily a subgroup of $G$.

We say that $G$ is **self-similar** if $G_v \leq G$ for every $v \in T$. It is **fractal/self-replicating** if $G_v = G$ for every $v \in T$.

Example. Gupta–Sidki $p$-group is fractal:
We can think of \( G_v \) as a subgroup of \( \text{Aut}(T) \) (after identifying \( T_v \) and \( T \)), but \( G_v \) is not necessarily a subgroup of \( G \).

We say that \( G \) is **self-similar** if \( G_v \leq G \) for every \( v \in T \).

It is **fractal/self-replicating** if \( G_v = G \) for every \( v \in T \).

Example. Gupta–Sidki \( p \)-group is fractal: \( \text{St}(1) = \langle b, b^a, \ldots, b^{a^{p-1}} \rangle \) where

\[
\begin{align*}
b & = (a, a^{-1}, 1, \ldots, 1, b), \\
b^a & = (b, a, a^{-1}, 1, \ldots, 1), \\
\ldots, \\
b^{a^{p-1}} & = (a^{-1}, 1, \ldots, 1, b, a)
\end{align*}
\]
We can think of $G_v$ as a subgroup of $\text{Aut}(T)$ (after identifying $T_v$ and $T$), but $G_v$ is not necessarily a subgroup of $G$.

We say that $G$ is **self-similar** if $G_v \leq G$ for every $v \in T$.

It is **fractal/self-replicating** if $G_v = G$ for every $v \in T$.

Example. Gupta–Sidki $p$-group is fractal: $\text{St}(1) = \langle b, b^a, \ldots, b^{a^{p-1}} \rangle$ where

\[
\begin{align*}
b &= (a, a^{-1}, 1, \ldots, 1, b), \\
b^a &= (b, a, a^{-1}, 1, \ldots, 1), \\
\ldots, \\
b^{a^{p-1}} &= (a^{-1}, 1, \ldots, 1, b, a)
\end{align*}
\]

Look at $v =$left-most vertex in first level; then $\varphi_v(b) = a, \varphi_v(b^a) = b$, so $G_v = G$.  

We can think of $G_\nu$ as a subgroup of $\text{Aut}(T)$ (after identifying $T_\nu$ and $T$), but $G_\nu$ is not necessarily a subgroup of $G$.

We say that $G$ is **self-similar** if $G_\nu \leq G$ for every $\nu \in T$. It is **fractal/self-replicating** if $G_\nu = G$ for every $\nu \in T$.

Example. Gupta–Sidki $p$-group is fractal: $\text{St}(1) = \langle b, b^a, \ldots, b^{ap-1} \rangle$ where

\[
b = (a, a^{-1}, 1, \ldots, 1, b),
\]

\[
b^a = (b, a, a^{-1}, 1, \ldots, 1),
\]

\[\ldots,\]

\[
b^{ap-1} = (a^{-1}, 1, \ldots, 1, b, a)
\]

Look at $\nu =$left-most vertex in first level; then $\varphi_\nu(b) = a, \varphi_\nu(b^a) = b$, so $G_\nu = G$. Similarly for rest of first level.

Look at $\nu =$left-most vertex in first level; then $\varphi_\nu(b) = a, \varphi_\nu(b^a) = b$, so $G_\nu = G$. Similarly for rest of first level.
We can think of $G_v$ as a subgroup of $\text{Aut}(T)$ (after identifying $T_v$ and $T$), but $G_v$ is not necessarily a subgroup of $G$.

We say that $G$ is **self-similar** if $G_v \leq G$ for every $v \in T$.

It is **fractal/self-replicating** if $G_v = G$ for every $v \in T$.

Example. Gupta–Sidki $p$-group is fractal: $\text{St}(1) = \langle b, b^a, \ldots, b^{a^{p-1}} \rangle$ where

$$b = (a, a^{-1}, 1, \ldots, 1, b),$$
$$b^a = (b, a, a^{-1}, 1, \ldots, 1),$$
$$\ldots,$$
$$b^{a^{p-1}} = (a^{-1}, 1, \ldots, 1, b, a)$$

Look at $v =$left-most vertex in first level; then $\varphi_v(b) = a, \varphi_v(b^a) = b$, so $G_v = G$. Similarly for rest of first level. **So we have St(1) subdirect in $G \times p$.**
We can think of $G_v$ as a subgroup of $\text{Aut}(T)$ (after identifying $T_v$ and $T$), but $G_v$ is not necessarily a subgroup of $G$.

We say that $G$ is **self-similar** if $G_v \leq G$ for every $v \in T$. It is **fractal/self-replicating** if $G_v = G$ for every $v \in T$.

Example. Gupta–Sidki $p$-group is fractal: $\text{St}(1) = \langle b, b^a, \ldots, b^{ap^{-1}} \rangle$ where

\[
\begin{align*}
b &= (a, a^{-1}, 1, \ldots, 1, b), \\
b^a &= (b, a, a^{-1}, 1, \ldots, 1), \\
\ldots, \\
b^{ap^{-1}} &= (a^{-1}, 1, \ldots, 1, b, a)
\end{align*}
\]

Look at $v =$left-most vertex in first level; then $\varphi_v(b) = a, \varphi_v(b^a) = b$, so $G_v = G$. Similarly for rest of first level. **So we have St(1) subdirect in $G \times p$.** Rest of vertices follow from $\varphi_u = \varphi_w \circ \varphi_v$ for $u = vw$. 
Self-similar results

Self-similarity/replication is very useful as it allows for length reduction arguments:

Write elements as words in generators, project using \( \phi \), usually get words of shorter length, still in \( G \).

Example (Grigorchuk, 1984) solvable word problem for 'spinal type' branch groups by a fast universal branch algorithm (Grigorchuk–Wilson, 2000) solvable conjugacy problem for wide class of branch groups (with some self-replication) (Bartholdi, 2003) every f.g. branch group with solvable word problem has finite "endomorphic" presentation

Question: Is there a f.p. branch/self-similar group?
Self-similar results

Self-similarity/replication is very useful as it allows for length reduction arguments:

- write elements as words in generators,
- project using $\varphi_v$,
- usually get words of shorter length, still in $G$.

Example (Grigorchuk, 1984) solvable word problem for 'spinal type' branch groups by a fast universal branch algorithm (Grigorchuk–Wilson, 2000) solvable conjugacy problem for wide class of branch groups (with some self-replication) (Bartholdi, 2003) every f.g. branch group with solvable word problem has finite "endomorphic" presentation

Question: Is there a f.p. branch/self-similar group?
Self-similar results

Self-similarity/replication is very useful as it allows for length reduction arguments:

- write elements as words in generators,
- project using $\varphi_v$,
- usually get words of shorter length, still in $G$.

Example

- (Grigorchuk, 1984) solvable word problem for ‘spinal type’ branch groups by a fast universal branch algorithm
Self-similar results

Self-similarity/replication is very useful as it allows for length reduction arguments:

- write elements as words in generators,
- project using $\varphi_v$,
- usually get words of shorter length, still in $G$.

Example

- (Grigorchuk, 1984) solvable word problem for ‘spinal type’ branch groups by a fast universal branch algorithm
- (Grigorchuk–Wilson, 2000) solvable conjugacy problem for wide class of branch groups (with some self-replication)
Self-similar results

Self-similarity/replication is very useful as it allows for length reduction arguments:

- write elements as words in generators,
- project using $\varphi_v$,
- usually get words of shorter length, still in $G$.

Example

- (Grigorchuk, 1984) solvable word problem for ‘spinal type’ branch groups by a fast universal branch algorithm
- (Grigorchuk–Wilson, 2000) solvable conjugacy problem for wide class of branch groups (with some self-replication)
- (Bartholdi, 2003) every f.g. branch group with solvable word problem has finite “endomorphic” presentation
Self-similar results

Self-similarity/replication is very useful as it allows for length reduction arguments:

- write elements as words in generators,
- project using $\varphi_v$,
- usually get words of shorter length, still in $G$.

Example

- (Grigorchuk, 1984) solvable word problem for ‘spinal type’ branch groups by a fast universal branch algorithm
- (Grigorchuk–Wilson, 2000) solvable conjugacy problem for wide class of branch groups (with some self-replication)
- (Bartholdi, 2003) every f.g. branch group with solvable word problem has finite “endomorphic” presentation

Question: Is there a f.p. branch/self-similar group?
More self-similar results

Take this even further:

**Theorem (G, 2013)**

Let \( G \) be the Gupta–Sidki 3-group. If \( H \leq G \) is finitely generated and infinite then there exists \( v \in T \) with \( H_v = G \).

Cfr: Theorem (Grigorchuk–Wilson, 2001)

All infinite finitely generated subgroups of the Grigorchuk group \( \Gamma \) are commensurable with \( \Gamma \).
More self-similar results

Take this even further:

**Theorem (G, 2013)**

Let $G$ be the Gupta–Sidki 3-group. If $H \leq G$ is finitely generated and infinite then there exists $v \in T$ with $H_v = G$.

This comes from (the proof of) an even stronger statement:

**Theorem 1 (G, 2013)**

If $H \leq G$ is finitely generated and infinite, then $H$ is (abstractly) commensurable with $G$ or $G \times G$.

Cfr: Theorem (Grigorchuk–Wilson, 2001) All infinite finitely generated subgroups of the Grigorchuk group $\Gamma$ are commensurable with $\Gamma$. 
More self-similar results

Take this even further:

**Theorem (G, 2013)**

Let $G$ be the Gupta–Sidki 3-group. If $H \leq G$ is finitely generated and infinite then there exists $v \in T$ with $H_v = G$.

This comes from (the proof of) an even stronger statement:

**Theorem 1 (G, 2013)**

If $H \leq G$ is finitely generated and infinite, then $H$ is (abstractly) commensurable with $G$ or $G \times G$.

Cfr:

**Theorem (Grigorchuk–Wilson, 2001)**

All infinite finitely generated subgroups of the Grigorchuk group $\Gamma$ are commensurable with $\Gamma$. 
Sketch proof

\( G = \text{Gupta–Sidki 3-group} \)
Gupta–Sidki 3-group

Auxiliary theorem

Let $\mathcal{X}$ be a class of subgroups of $G$ satisfying

1. $1, G \in \mathcal{X}$
2. closed for finite index supergroups
3. if all first level projections of $H$ are in $\mathcal{X}$ then so is $H$.

Then $\mathcal{X}$ contains all finitely generated subgroups of $G$. 

Proof: By contradiction; length reduction argument.

Take $H \leq G$ finitely generated, $H/\in \mathcal{X}$ with "shortest" generating set.

Then $H/v/ \in \mathcal{X}$ for every $v$ in first level, so $H/u/ \in \mathcal{X}$ for some $u$ in second level.

Technical work, to get that $H/u$ has a shorter generating set than $H$.

The "technical work" only works for $p = 3$; everything else works for all odd primes.

Alejandra Garrido (Oxford)
Groups that look like trees
GTI Webinar, Dec 2014
Sketch proof

$G =$Gupta–Sidki 3-group

Auxiliary theorem

Let $\mathcal{X}$ be a class of subgroups of $G$ satisfying

1. $1, G \in \mathcal{X}$
2. closed for finite index supergroups
3. if all first level projections of $H$ are in $\mathcal{X}$ then so is $H$.

Then $\mathcal{X}$ contains all finitely generated subgroups of $G$.

Proof: By contradiction; length reduction argument.
Sketch proof

\[ G = \text{Gupta–Sidki 3-group} \]

**Auxiliary theorem**

Let \( \mathcal{X} \) be a class of subgroups of \( G \) satisfying

1. \( 1, G \in \mathcal{X} \)
2. closed for finite index supergroups
3. if all first level projections of \( H \) are in \( \mathcal{X} \) then so is \( H \).

Then \( \mathcal{X} \) contains all finitely generated subgroups of \( G \).

Proof: By contradiction; length reduction argument. Take \( H \leq G \) finitely generated, \( H \notin \mathcal{X} \) with “shortest” generating set.
Sketch proof

\( G = \text{Gupta–Sidki 3-group} \)

**Auxiliary theorem**

Let \( \mathcal{X} \) be a class of subgroups of \( G \) satisfying

1. \( 1, G \in \mathcal{X} \)
2. closed for finite index supergroups
3. if all first level projections of \( H \) are in \( \mathcal{X} \) then so is \( H \).

Then \( \mathcal{X} \) contains all finitely generated subgroups of \( G \).

Proof: By contradiction; length reduction argument. Take \( H \leq G \) finitely generated, \( H \notin \mathcal{X} \) with “shortest” generating set. Then \( H_v \notin \mathcal{X} \) for every \( v \) in first level, so \( H_u \notin \mathcal{X} \) for some \( u \) in second level.
Sketch proof

\[ G = \text{Gupta–Sidki 3-group} \]

**Auxiliary theorem**

Let \( \mathcal{X} \) be a class of subgroups of \( G \) satisfying

1. \( 1, G \in \mathcal{X} \)
2. closed for finite index supergroups
3. if all first level projections of \( H \) are in \( \mathcal{X} \) then so is \( H \).

Then \( \mathcal{X} \) contains all finitely generated subgroups of \( G \).

**Proof:** By contradiction; length reduction argument. Take \( H \leq G \) finitely generated, \( H \notin \mathcal{X} \) with “shortest” generating set. Then \( H_v \notin \mathcal{X} \) for every \( v \) in first level, so \( H_u \notin \mathcal{X} \) for some \( u \) in second level. Technical work, to get that \( H_u \) has a shorter generating set than \( H \).
Sketch proof

\( G = \text{Gupta–Sidki 3-group} \)

**Auxiliary theorem**

Let \( \mathcal{X} \) be a class of subgroups of \( G \) satisfying:

1. \( 1, G \in \mathcal{X} \)
2. closed for finite index supergroups
3. if all first level projections of \( H \) are in \( \mathcal{X} \) then so is \( H \).

Then \( \mathcal{X} \) contains all finitely generated subgroups of \( G \).

Proof: By contradiction; length reduction argument. Take \( H \leq G \) finitely generated, \( H \notin \mathcal{X} \) with “shortest” generating set. Then \( H_v \notin \mathcal{X} \) for every \( v \) in first level, so \( H_u \notin \mathcal{X} \) for some \( u \) in second level. Technical work, to get that \( H_u \) has a shorter generating set than \( H \).

The “technical work” only works for \( p = 3 \); everything else works for all odd primes.
Auxiliary Theorem

Let $\mathcal{X}$ be a class of subgroups of $G$ satisfying

1. $1, G \in \mathcal{X}$
2. closed for finite index supergroups
3. if all first level projections of $H$ are in $\mathcal{X}$ then so is $H$.

Then $\mathcal{X}$ contains all finitely generated subgroups of $G$. 

1 & 2: easy
3: $H$ is subdirect product of its projections $H_1, H_2, H_3$.
Each $H_i$ is commensurable with $G$ or $G \times G$.

Now, use fact that each finite index subgroup of $G$ contains some $\text{St}(n)$ (congruence subgroup property, more to follow) and $\text{St}(n)$ is subdirect in $G \times 3^n$.
Reduce to $H$ subdirect in $H_1 \times G \times k$, of finite index because $G$ is just infinite.
Finish using fact that $G \times i$ and $G \times j$ are commensurable if $i \equiv j \mod 2$. 

Alejandra Garrido (Oxford) Groups that look like trees GTI Webinar, Dec 2014 12 / 26
Auxiliary Theorem

Let $\mathcal{X}$ be a class of subgroups of $G$ satisfying

1. $1, G \in \mathcal{X}$
2. closed for finite index supergroups
3. if all first level projections of $H$ are in $\mathcal{X}$ then so is $H$.

Then $\mathcal{X}$ contains all finitely generated subgroups of $G$.

So suffices to show that $\mathcal{C}(\text{:=subgroups of } G \text{ which are finite, or commensurable with } G \text{ or } G \times G)$ satisfies 3 conditions in Auxiliary Theorem.
Auxiliary Theorem

Let $\mathcal{X}$ be a class of subgroups of $G$ satisfying

1. $G \in \mathcal{X}$
2. closed for finite index supergroups
3. if all first level projections of $H$ are in $\mathcal{X}$ then so is $H$.

Then $\mathcal{X}$ contains all finitely generated subgroups of $G$.

So suffices to show that $C(\text{:=subgroups of } G \text{ which are finite, or commensurable with } G \text{ or } G \times G)$ satisfies 3 conditions in Auxiliary Theorem. 1 & 2: easy
Auxiliary Theorem

Let $\mathcal{X}$ be a class of subgroups of $G$ satisfying

1. $G \in \mathcal{X}$
2. closed for finite index supergroups
3. if all first level projections of $H$ are in $\mathcal{X}$ then so is $H$.

Then $\mathcal{X}$ contains all finitely generated subgroups of $G$.

So suffices to show that $\mathcal{C}(:=\text{subgroups of } G \text{ which are finite, or commensurable with } G \text{ or } G \times G)$ satisfies 3 conditions in Auxiliary Theorem. 1 & 2: easy
3: $H$ is subdirect product of its projections $H_1, H_2, H_3$. 
Auxiliary Theorem

Let $\mathcal{X}$ be a class of subgroups of $G$ satisfying

1. $1, G \in \mathcal{X}$
2. closed for finite index supergroups
3. if all first level projections of $H$ are in $\mathcal{X}$ then so is $H$.

Then $\mathcal{X}$ contains all finitely generated subgroups of $G$.

So suffices to show that $\mathcal{C}(:=\text{subgroups of } G \text{ which are finite, or commensurable with } G \text{ or } G \times G)$ satisfies 3 conditions in Auxiliary Theorem. 1 & 2: easy

3. $H$ is subdirect product of its projections $H_1, H_2, H_3$. Each $H_i$ is commensurable with $G$ or $G \times G$. 

Alejandra Garrido (Oxford) 
Groups that look like trees 
GTI Webinar, Dec 2014
Auxiliary Theorem

Let $\mathcal{X}$ be a class of subgroups of $G$ satisfying

1. $1, G \in \mathcal{X}$
2. closed for finite index supergroups
3. if all first level projections of $H$ are in $\mathcal{X}$ then so is $H$.

Then $\mathcal{X}$ contains all finitely generated subgroups of $G$.

So suffices to show that $\mathcal{C}(\text{subgroups of } G \text{ which are finite, or commensurable with } G \text{ or } G \times G)$ satisfies 3 conditions in Auxiliary Theorem. 1 & 2: easy

3: $H$ is subdirect product of its projections $H_1, H_2, H_3$. Each $H_i$ is commensurable with $G$ or $G \times G$. Now, use fact that each finite index subgroup of $G$ contains some $\text{St}(n)$ (congruence subgroup property, more to follow)
Auxiliary Theorem

Let $\mathcal{X}$ be a class of subgroups of $G$ satisfying

1. $1, G \in \mathcal{X}$
2. closed for finite index supergroups
3. if all first level projections of $H$ are in $\mathcal{X}$ then so is $H$.

Then $\mathcal{X}$ contains all finitely generated subgroups of $G$.

So suffices to show that $\mathcal{C}(:=\text{subgroups of } G \text{ which are finite, or commensurable with } G \text{ or } G \times G)$ satisfies 3 conditions in Auxiliary Theorem. 1 & 2: easy

3: $H$ is subdirect product of its projections $H_1, H_2, H_3$. Each $H_i$ is commensurable with $G$ or $G \times G$. Now, use fact that each finite index subgroup of $G$ contains some $\text{St}(n)$ (congruence subgroup property, more to follow) and $\text{St}(n)$ is subdirect in $G^{\times 3^n}$. 
Auxiliary Theorem

Let $\mathcal{X}$ be a class of subgroups of $G$ satisfying

1. $1, G \in \mathcal{X}$
2. closed for finite index supergroups
3. if all first level projections of $H$ are in $\mathcal{X}$ then so is $H$.

Then $\mathcal{X}$ contains all finitely generated subgroups of $G$.

So suffices to show that $\mathcal{C}(:=\text{subgroups of } G \text{ which are finite, or commensurable with } G \text{ or } G \times G)$ satisfies 3 conditions in Auxiliary Theorem. 1 & 2: easy

3: $H$ is subdirect product of its projections $H_1, H_2, H_3$. Each $H_i$ is commensurable with $G$ or $G \times G$. Now, use fact that each finite index subgroup of $G$ contains some $\text{St}(n)$ (congruence subgroup property, more to follow) and $\text{St}(n)$ is subdirect in $G^{\times 3^n}$. Reduce to $H$ subdirect in $H_1 \times G^{\times k}$,
Auxiliary Theorem

Let $\mathcal{X}$ be a class of subgroups of $G$ satisfying

1. $G \in \mathcal{X}$
2. closed for finite index supergroups
3. if all first level projections of $H$ are in $\mathcal{X}$ then so is $H$.

Then $\mathcal{X}$ contains all finitely generated subgroups of $G$.

So suffices to show that $\mathcal{C}(\text{:=subgroups of } G \text{ which are finite, or commensurable with } G \text{ or } G \times G)$ satisfies 3 conditions in Auxiliary Theorem. 1 & 2: easy

3: $H$ is subdirect product of its projections $H_1, H_2, H_3$. Each $H_i$ is commensurable with $G$ or $G \times G$. Now, use fact that each finite index subgroup of $G$ contains some $\text{St}(n)$ (congruence subgroup property, more to follow) and $\text{St}(n)$ is subdirect in $G^{\times 3^n}$. Reduce to $H$ subdirect in $H_1 \times G^{\times k}$, of finite index because $G$ is just infinite.
Auxiliary Theorem

Let $\mathcal{X}$ be a class of subgroups of $G$ satisfying

1. $1, G \in \mathcal{X}$
2. closed for finite index supergroups
3. if all first level projections of $H$ are in $\mathcal{X}$ then so is $H$.

Then $\mathcal{X}$ contains all finitely generated subgroups of $G$.

So suffices to show that $\mathcal{C}$ (subgroups of $G$ which are finite, or commensurable with $G$ or $G \times G$) satisfies 3 conditions in Auxiliary Theorem. 1 & 2: easy

3. $H$ is subdirect product of its projections $H_1, H_2, H_3$. Each $H_i$ is commensurable with $G$ or $G \times G$. Now, use fact that each finite index subgroup of $G$ contains some $\text{St}(n)$ (congruence subgroup property, more to follow) and $\text{St}(n)$ is subdirect in $G^\times 3^n$. Reduce to $H$ subdirect in $H_1 \times G^\times k$, of finite index because $G$ is just infinite. Finish using fact that $G^\times i$ and $G^\times j$ are commensurable if $i \equiv j \mod 2$. 
Similarly, use Auxiliary Theorem and Theorem 1 to prove

**Theorem (G, 2013)**

*G is subgroup separable (LERF), i.e., all finitely generated subgroups are an intersection of finite index subgroups.*
Similarly, use Auxiliary Theorem and Theorem 1 to prove

**Theorem (G, 2013)**

*G is subgroup separable (LERF), i.e., all finitely generated subgroups are an intersection of finite index subgroups.*

Remains to show that $G$ and $G \times G$ are not commensurable.
Similarly, use Auxiliary Theorem and Theorem 1 to prove

**Theorem (G, 2013)**

$G$ is subgroup separable (LERF), i.e., all finitely generated subgroups are an intersection of finite index subgroups.

Remains to show that $G$ and $G \times G$ are not commensurable. Idea: write subgroups of finite index as subdirect products; look at the number of factors. Need to know about normal subgroups of subgroups of finite index.
Similarly, use Auxiliary Theorem and Theorem 1 to prove

**Theorem (G, 2013)**

*G* is subgroup separable (LERF), i.e., all finitely generated subgroups are an intersection of finite index subgroups.

Remains to show that *G* and *G* × *G* are not commensurable. Idea: write subgroups of finite index as subdirect products; look at the number of factors. Need to know about normal subgroups of subgroups of finite index. Second way in which these groups look like trees...
1. Introduction

2. Self-similarity

3. Branch structure
$T$ = rooted tree of type $(m_n)_n$. $G$ acts faithfully on $T$.

**Definition**

$rst_G(v) := \{ g \in G : g \text{ fixes all vertices outside } T_v \}$ is the rigid stabilizer of $v \in T$.

$rst_G(n) := \prod_{v \in V_n} rst_G(v)$ is the rigid stabilizer of level $n$. 

$T_v$
**Definition**

$G$ acts as a branch group on $T$ iff for every $n$:

1. $G$ acts transitively on $V_n$ (‘acts level-transitively on $T$’)
2. $|G : \text{rst}_G(n)| < \infty$

---

**Examples**

- For all $n$, $A = \text{Aut}(T)$ acts transitively on $V_n$ with kernel $\text{rst}_A(n)$.
- Gupta–Sidki $p$-groups
- Grigorchuk groups
- Aleshin group
Branch group: definition

**Definition**

$G$ acts as a branch group on $T$ iff for every $n$:

1. $G$ acts transitively on $V_n$ (‘acts level-transitively on $T$’)
2. $|G : \text{rst}_G(n)| < \infty$

**Definition**

$G$ is branch if it acts faithfully as a branch group on some $T$. 
Definition

$G$ acts as a branch group on $T$ iff for every $n$:

1. $G$ acts transitively on $V_n$ (‘acts level-transitively on $T$’)
2. $|G : \text{rst}_G(n)| < \infty$

Definition

$G$ is branch if it acts faithfully as a branch group on some $T$.

Examples

- For all $n$, $A = \text{Aut}(T)$ acts transitively on $V_n$ with kernel $\text{rst}_A(n)$.
- Gupta–Sidki $p$-groups
- Grigorchuk groups
- Aleshin group
Key lemma (Grigorchuk)

If $G$ is branch and $1 \neq K \triangleleft G$ then $\text{rst}_G(n)' \leq K$ for some $n$. 

Theorem 2 (G–Wilson, 2014)

Let $G$ branch, $1 \neq K \triangleleft H \leq f G$. For all $n$ sufficiently large, $K \cap \text{rst}_G(n)' = \text{rst}_G(X)'$ where $X$ is a union of orbits of $H$ at level $n$. 

We can use this to give an isomorphism invariant for $H$: 
Subgroups of branch groups

Key lemma (Grigorchuk)
If $G$ is branch and $1 \neq K \triangleleft G$ then $\text{rst}_G(n)' \leq K$ for some $n$.

Theorem 2 (G–Wilson, 2014)
Let $G$ branch, $1 \neq K \triangleleft H \leq_f G$. For all $n$ sufficiently large,

$$K \cap \text{rst}_G(n)' = \text{rst}_G(X)'$$

where $X$ is a union of orbits of $H$ at level $n$. 

Alejandra Garrido (Oxford)
Groups that look like trees
GTI Webinar, Dec 2014
Subgroups of branch groups

Key lemma (Grigorchuk)
If $G$ is branch and $1 \neq K \triangleleft G$ then $\text{rst}_G(n)' \leq K$ for some $n$.

Theorem 2 (G–Wilson, 2014)
Let $G$ branch, $1 \neq K \triangleleft H \leq_f G$. For all $n$ sufficiently large,

$$K \cap \text{rst}_G(n)' = \text{rst}_G(X)'$$

where $X$ is a union of orbits of $H$ at level $n$.

We can use this to give an isomorphism invariant for $H$: 
Theorem 2 (G–Wilson, 2014)

Let $G$ branch, $1 \neq K \triangleleft H \leq_f G$. For all $n$ sufficiently large,

$$K \cap \text{rst}_G(n)' = \text{rst}_G(X)'$$

where $X$ is a union of orbits of $H$ at level $n$. 
Theorem 2 (G–Wilson, 2014)

Let $G$ branch, $1 \not= K \triangleleft H \leq_f G$. For all $n$ sufficiently large,

$$K \cap \text{rst}_G(n)' = \text{rst}_G(X)'$$

where $X$ is a union of orbits of $H$ at level $n$.

$b(H) :=$ maximum number of infinite normal subgroups of $H$ that generate their direct product.
Finite index subgroups of branch groups

Theorem 2 (G–Wilson, 2014)

Let $G$ branch, $1 \neq K \triangleleft H \leq_{f} G$. For all $n$ sufficiently large,

$$K \cap \text{rst}_{G}(n)' = \text{rst}_{G}(X)'$$

where $X$ is a union of orbits of $H$ at level $n$.

$b(H) :=$ maximum number of infinite normal subgroups of $H$ that generate their direct product. By Theorem 2, $b(H) \leq$ maximum number of orbits of $H$ on any layer of $T$. 
Theorem 2 (G–Wilson, 2014)

Let $G$ branch, $1 \neq K \triangleleft H \leq_f G$. For all $n$ sufficiently large,

$$K \cap \text{rst}_G(n)' = \text{rst}_G(X)'$$

where $X$ is a union of orbits of $H$ at level $n$.

$b(H) := \text{maximum number of infinite normal subgroups of } H \text{ that generate their direct product.}$ By Theorem 2, $b(H) \leq \text{maximum number of orbits of } H \text{ on any layer of } T$. The number of $H$-orbits on any layer is bounded (by $|G:H|$).

Say $V_n = X_1 \sqcup \ldots \sqcup X_r$, each $X_i$ an $H$-orbit.

Then $\text{rst}_G(X_i)' \triangleleft H$ and $\text{rst}_G(n)' = \prod \text{rst}_G(X_i)' \triangleleft H$. 

Finite index subgroups of branch groups

Theorem 2 (G–Wilson, 2014)

Let $G$ branch, $1 \neq K \triangleleft H \leq_f G$. For all $n$ sufficiently large,

$$K \cap \text{rst}_G(n)' = \text{rst}_G(X)'$$

where $X$ is a union of orbits of $H$ at level $n$.

$b(H) :=$ maximum number of infinite normal subgroups of $H$ that generate their direct product. By Theorem 2, $b(H) \leq$ maximum number of orbits of $H$ on any layer of $T$. The number of $H$-orbits on any layer is bounded (by $|G : H|$).

Say $V_n = X_1 \sqcup \ldots \sqcup X_r$, each $X_i$ an $H$-orbit.

Then $\text{rst}_G(X_i)' \triangleleft H$ and $\text{rst}_G(n)' = \prod \text{rst}_G(X_i)' \triangleleft H$.

Corollary

$b(H) =$ maximum number of orbits of $H$ on any layer of $T$. 
How it all fits together

\( b(H) \) behaves well under direct products

Let \( H \leq_f H_1 \times \ldots \times H_r \) be subdirect; \( b(H_i) \) finite. Then \( b(H) = b(H_1) + \ldots + b(H_r) \).
How it all fits together

**b(H) behaves well under direct products**

Let $H \leq_f H_1 \times \ldots \times H_r$ be subdirect; $b(H_i)$ finite. Then $b(H) = b(H_1) + \ldots + b(H_r)$.

**Easy lemma**

Let $H \leq_f G$ act like a $p$-group on every layer of the $p$-regular tree. Then $b(H) \equiv 1 \mod p - 1$.
How it all fits together

**b(H) behaves well under direct products**

Let $H \leq_f H_1 \times \ldots \times H_r$ be subdirect; $b(H_i)$ finite.
Then $b(H) = b(H_1) + \ldots + b(H_r)$.

**Easy lemma**

Let $H \leq_f G$ act like a $p$-group on every layer of the $p$-regular tree.
Then $b(H) \equiv 1 \mod p - 1$.

**Corollary**

Let $\Gamma_1, \Gamma_2$ be direct products of $n_1, n_2$ branch groups acting like $p$-groups on every layer of the $p$-regular tree.
If $\Gamma_1$ and $\Gamma_2$ are commensurable, then $n_1 \equiv n_2 \mod p - 1$. 
How it all fits together

**b(H) behaves well under direct products**

Let $H \leq_f H_1 \times \ldots \times H_r$ be subdirect; $b(H_i)$ finite. Then $b(H) = b(H_1) + \ldots + b(H_r)$.

**Easy lemma**

Let $H \leq_f G$ act like a $p$-group on every layer of the $p$-regular tree. Then $b(H) \equiv 1 \pmod{p-1}$.

**Corollary**

Let $\Gamma_1, \Gamma_2$ be direct products of $n_1, n_2$ branch groups acting like $p$-groups on every layer of the $p$-regular tree. If $\Gamma_1$ and $\Gamma_2$ are commensurable, then $n_1 \equiv n_2 \pmod{p-1}$.

So the Gupta–Sidki 3-group has 3 commensurability classes of f.g. subgroups.
Old idea of Wilson for classification of just infinite groups (a group is just non-$P$ if it is not $P$ but all its proper quotients are $P$).
Old idea of Wilson for classification of just infinite groups (a group is just non-$P$ if it is not $P$ but all its proper quotients are $P$). One of the classes is that of just infinite branch groups.
Old idea of Wilson for classification of just infinite groups (a group is just non-$P$ if it is not $P$ but all its proper quotients are $P$). One of the classes is that of just infinite branch groups.

- Look at subnormal subgroups with finitely many conjugates of just infinite groups
Old idea of Wilson for classification of just infinite groups (a group is just non-$P$ if it is not $P$ but all its proper quotients are $P$). One of the classes is that of just infinite branch groups.

- Look at subnormal subgroups with finitely many conjugates of just infinite groups
- quotient by commensurability
Old idea of Wilson for classification of just infinite groups (a group is just non-$P$ if it is not $P$ but all its proper quotients are $P$). One of the classes is that of just infinite branch groups.

- Look at subnormal subgroups with finitely many conjugates of just infinite groups
- Quotient by commensurability
- Obtain structure lattice.
Old idea of Wilson for classification of just infinite groups (a group is just non-$P$ if it is not $P$ but all its proper quotients are $P$). One of the classes is that of just infinite branch groups.

- Look at subnormal subgroups with finitely many conjugates of just infinite groups
- quotient by commensurability
- obtain structure lattice.

 Turns out we only need to look at subgroups with finitely many conjugates.
\[ L(G) := \{ K \mid K \triangleleft H \leq_f G \} \]
Structure lattice

\[ L(G) := \{ K \mid K \triangleleft H \leq_f G \} \]

By Key Lemma, all branch groups are just non-(virtually abelian).
Structure lattice

\[ L(G) := \{ K \mid K \triangleleft H \leq_{f} G \} \]

By Key Lemma, all branch groups are just non-(virtually abelian).

\[ K_1 \sim K_2 \text{ iff } K_1/(K_1 \cap K_2), K_2/(K_1 \cap K_2) \text{ are virtually abelian.} \]
\( L(G) := \{ K \mid K \triangleleft H \leq_f G \} \)

By Key Lemma, all branch groups are just non-(virtually abelian).

\( K_1 \sim K_2 \iff \frac{K_1}{(K_1 \cap K_2)}, \frac{K_2}{(K_1 \cap K_2)} \) are virtually abelian.

\( L := L(G)/\sim \) is a lattice:

\[ [K_1] \wedge [K_2] = [K_1 \cap K_2], \]

\[ [K_1] \vee [K_2] = [\langle K_1, K_2 \rangle], \] order induced by subgroup inclusion.
Structure lattice

\[ L(G) := \{K \mid K \triangleleft H \leq_f G\} \]

By Key Lemma, all branch groups are just non-(virtually abelian).

\[ K_1 \sim K_2 \text{ iff } K_1/(K_1 \cap K_2), \ K_2/(K_1 \cap K_2) \text{ are virtually abelian.} \]

\[ \mathcal{L} := L(G)/\sim \text{ is a lattice: } [K_1] \wedge [K_2] = [K_1 \cap K_2], \]
\[ [K_1] \vee [K_2] = [\langle K_1, K_2 \rangle], \text{ order induced by subgroup inclusion.} \]

**Definition**

\( \mathcal{L} \) is the **structure lattice** of \( G \).
Structure lattice

\[ L(G) := \{ K \mid K \triangleleft H \leq_f G \} \]

By Key Lemma, all branch groups are just non-(virtually abelian).

\( K_1 \sim K_2 \) iff \( K_1/(K_1 \cap K_2), K_2/(K_1 \cap K_2) \) are virtually abelian.

\[ \mathcal{L} := L(G)/\sim \text{ is a lattice: } [K_1] \wedge [K_2] = [K_1 \cap K_2], \]
\[ [K_1] \vee [K_2] = [\langle K_1, K_2 \rangle], \text{ order induced by subgroup inclusion.} \]

**Definition**

\( \mathcal{L} \) is the **structure lattice** of \( G \).

Conjugation by \( G \) induces a well-defined action of \( G \) on \( \mathcal{L} \).
Structure lattice

\[ L(G) := \{ K \mid K \triangleleft H \leq_f G \} \]

By Key Lemma, all branch groups are just non-(virtually abelian).
\[ K_1 \sim K_2 \text{ iff } K_1/(K_1 \cap K_2), K_2/(K_1 \cap K_2) \text{ are virtually abelian.} \]

\[ \mathcal{L} := L(G)/\sim \text{ is a lattice: } [K_1] \land [K_2] = [K_1 \cap K_2], \]
\[ [K_1] \lor [K_2] = [\langle K_1, K_2 \rangle], \text{ order induced by subgroup inclusion.} \]

**Definition**

\[ \mathcal{L} \text{ is the structure lattice of } G. \]

Conjugation by \( G \) induces a well-defined action of \( G \) on \( \mathcal{L} \).
So, reformulating, we have

**Theorem 2**

Every element of \( \mathcal{L} \) has as a representative some \( \text{rst}(X) \) where \( X \) is an \( H \)-orbit for some \( H \leq_f G \).
By analogy with the classical case of linear algebraic groups, we have

**Definition**

A group $G$ acting faithfully on a rooted tree has the **congruence subgroup property (CSP)** if for every $H \leq_{f} G$ there is some $n$ with $\text{St}(n) \leq H$.

**Example:** Gupta–Sidki $p$-groups (used in proof of Theorem 1).

**Question** (Bartholdi–Siegenthaler–Zalesskii, 2012)

For a branch group, does having CSP depend on the chosen branch action?

**No!**

**Theorem 3** (G, 2014)

Whether a branch group has CSP or not is independent of the branch action.
Application: Congruence subgroup property

By analogy with the classical case of linear algebraic groups, we have

**Definition**

A group $G$ acting faithfully on a rooted tree has the **congruence subgroup property (CSP)** if for every $H \leq_f G$ there is some $n$ with $\text{St}(n) \leq H$.

Example: Gupta–Sidki $p$-groups (used in proof of Theorem 1).
By analogy with the classical case of linear algebraic groups, we have

**Definition**

A group $G$ acting faithfully on a rooted tree has the congruence subgroup property (CSP) if for every $H \leq_f G$ there is some $n$ with $\text{St}(n) \leq H$.

Example: Gupta–Sidki $p$-groups (used in proof of Theorem 1).

**Question (Bartholdi–Siegenthaler–Zalesskii, 2012)**

For a branch group, does having CSP depend on the chosen branch action?
By analogy with the classical case of linear algebraic groups, we have

**Definition**

A group $G$ acting faithfully on a rooted tree has the **congruence subgroup property (CSP)** if for every $H \leq_f G$ there is some $n$ with $St(n) \leq H$.

**Example:** Gupta–Sidki $p$-groups (used in proof of Theorem 1).

**Question (Bartholdi–Siegenthaler–Zalesskii, 2012)**

For a branch group, does having CSP depend on the chosen branch action?

No!
By analogy with the classical case of linear algebraic groups, we have

**Definition**

A group $G$ acting faithfully on a rooted tree has the **congruence subgroup property (CSP)** if for every $H \leq_f G$ there is some $n$ with $\text{St}(n) \leq H$.

Example: Gupta–Sidki $p$-groups (used in proof of Theorem 1).

**Question (Bartholdi–Siegenthaler–Zalesskii, 2012)**

For a branch group, does having CSP depend on the chosen branch action?

No!

**Theorem 3 (G, 2014)**

Whether a branch group has CSP or not is independent of the branch action.
Theorem 2

Every element of $\mathcal{L}$ has as a representative some $\text{rst}(X)$ where $X$ is an $H$-orbit for some $H \leq_f G$.
Theorem 2

Every element of $\mathcal{L}$ has as a representative some $\text{rst}(X)$ where $X$ is an $H$-orbit for some $H \leq_f G$.

In particular, for any branch action $\rho : G \to \text{Aut}(T_\rho)$ and any $[K] \in \mathcal{L}$ there exists $v \in T_\rho$ with $[K] \geq [\text{rst}_\rho(v)]$. 
Proof ingredients

**Theorem 2**

Every element of $\mathcal{L}$ has as a representative some $\text{rst}(X)$ where $X$ is an $H$-orbit for some $H \leq_f G$.

In particular, for any branch action $\rho : G \to \text{Aut}(T_\rho)$ and any $[K] \in \mathcal{L}$ there exists $v \in T_\rho$ with $[K] \geq [\text{rst}_\rho(v)]$.

**Lemma**

If $G$ acts as a branch group on $T$ then $T$ embeds $G$-equivariantly in $\mathcal{L}$: $v \mapsto [\text{rst}_G(v)]$. 
Proof

To show that having CSP is independent of the branch action, we need to show that given two branch actions $\sigma : G \rightarrow \text{Aut}(T_\sigma)$ and $\rho : G \rightarrow \text{Aut}(T_\rho)$ every $\text{St}_\sigma(n)$ contains some $\text{St}_\rho(m)$ and vice-versa.

Let $u \in T_\sigma$ of level $n$. By the above, there is some $v \in T_\rho$ (call its level $m$) such that $\text{rst}_\sigma(u) \geq \text{rst}_\rho(v)$. Now, if $x \in \text{St}_\rho(m)$, we have $1 \neq \text{rst}_\rho(v) \leq \text{rst}_\sigma(u) \land \text{rst}_\sigma(u) = \text{rst}_\sigma(ux) \cap \text{rst}_\sigma(u)$, so $\text{rst}_\sigma(ux) = \text{rst}_\sigma(u)$.

Hence $x \in \text{St}_\sigma(u)$. To finish, use transitivity of $G$ on all levels of $T_\rho$ and $T_\sigma$ to get $x \in \bigcap g \in G \text{St}_\sigma(ug) = \text{St}_\sigma(n)$.
Proof

To show that having CSP is independent of the branch action, we need to show that given two branch actions \( \sigma : G \to \text{Aut}(T_\sigma) \) and \( \rho : G \to \text{Aut}(T_\rho) \) every \( \text{St}_\sigma(n) \) contains some \( \text{St}_\rho(m) \) and vice-versa.

- Take \( u \in T_\sigma \) of level \( n \).

\[ r^\text{st}_\sigma(u) \geq r^\text{st}_\rho(v) \] where \( v \) is some element in \( T_\rho \) of level \( m \) such that \( u \) and \( v \) have the same level.

Now, if \( x \in \text{St}_\rho(m) \), we have

\[ 1 \neq r^\text{st}_\rho(v) \leq r^\text{st}_\sigma(u) \]

\[ x \land r^\text{st}_\sigma(u) = r^\text{st}_\sigma(ux) \cap r^\text{st}_\sigma(u) \]

Hence \( x \in \text{St}_\sigma(u) \).

To finish, use transitivity of \( G \) on all levels of \( T_\rho \) and \( T_\sigma \) to get

\[ x \in \bigcap_{g \in G} \text{St}_\sigma(ug) = \text{St}_\sigma(n) \].
Proof

To show that having CSP is independent of the branch action, we need to show that given two branch actions $\sigma : G \to \text{Aut}(T_\sigma)$ and $\rho : G \to \text{Aut}(T_\rho)$ every $\text{St}_\sigma(n)$ contains some $\text{St}_\rho(m)$ and vice-versa.

- Take $u \in T_\sigma$ of level $n$.
- By the above, there is some $v \in T_\rho$ (call its level $m$) such that $[\text{rst}_\sigma(u)] \geq [\text{rst}_\rho(v)]$.
Proof

To show that having CSP is independent of the branch action, we need to show that given two branch actions \( \sigma : G \to \text{Aut}(T_\sigma) \) and \( \rho : G \to \text{Aut}(T_\rho) \) every \( \text{St}_\sigma(n) \) contains some \( \text{St}_\rho(m) \) and vice-versa.

- Take \( u \in T_\sigma \) of level \( n \).
- By the above, there is some \( v \in T_\rho \) (call its level \( m \)) such that \( \text{rst}_\sigma(u) \geq \text{rst}_\rho(v) \).
- Now, if \( x \in \text{St}_\rho(m) \), we have

\[
1 \neq \text{rst}_\rho(v) \leq \text{rst}_\sigma(u)^x \land \text{rst}_\sigma(u) = \text{rst}_\sigma(ux) \cap \text{rst}_\sigma(u),
\]

so \( \text{rst}_\sigma(ux) = \text{rst}_\sigma(u) \).
Proof

To show that having CSP is independent of the branch action, we need to show that given two branch actions $\sigma : G \to \text{Aut}(T_\sigma)$ and $\rho : G \to \text{Aut}(T_\rho)$ every $\text{St}_\sigma(n)$ contains some $\text{St}_\rho(m)$ and vice-versa.

- Take $u \in T_\sigma$ of level $n$.
- By the above, there is some $v \in T_\rho$ (call its level $m$) such that $[\text{rst}_\sigma(u)] \geq [\text{rst}_\rho(v)]$.
- Now, if $x \in \text{St}_\rho(m)$, we have
  \[ 1 \neq [\text{rst}_\rho(v)] \leq [\text{rst}_\sigma(u)]^x \land [\text{rst}_\sigma(u)] = [\text{rst}_\sigma(ux) \cap \text{rst}_\sigma(u)], \]

  so $\text{rst}_\sigma(ux) = \text{rst}_\sigma(u)$.
- Hence $x \in \text{St}_\sigma(u)$. 

Alejandra Garrido (Oxford)  
Groups that look like trees  
GTI Webinar, Dec 2014  
24 / 26
Proof

To show that having CSP is independent of the branch action, we need to show that given two branch actions $\sigma : G \to \text{Aut}(T_\sigma)$ and $\rho : G \to \text{Aut}(T_\rho)$ every $\text{St}_\sigma(n)$ contains some $\text{St}_\rho(m)$ and vice-versa.

- Take $u \in T_\sigma$ of level $n$.
- By the above, there is some $v \in T_\rho$ (call its level $m$) such that $[\text{rst}_\sigma(u)] \geq [\text{rst}_\rho(v)]$.
- Now, if $x \in \text{St}_\rho(m)$, we have
  \[
  1 \neq [\text{rst}_\rho(v)] \leq [\text{rst}_\sigma(u)]^x \land [\text{rst}_\sigma(u)] = [\text{rst}_\sigma(ux) \cap \text{rst}_\sigma(u)],
  \]
  so $\text{rst}_\sigma(ux) = \text{rst}_\sigma(u)$.
- Hence $x \in \text{St}_\sigma(u)$.
- To finish, use transitivity of $G$ on all levels of $T_\rho$ and $T_\sigma$ to get
To show that having CSP is independent of the branch action, we need to show that given two branch actions $\sigma : G \to \text{Aut}(T_\sigma)$ and $\rho : G \to \text{Aut}(T_\rho)$ every $\text{St}_\sigma(n)$ contains some $\text{St}_\rho(m)$ and vice-versa.

- Take $u \in T_\sigma$ of level $n$.
- By the above, there is some $v \in T_\rho$ (call its level $m$) such that $[\text{rst}_\sigma(u)] \geq [\text{rst}_\rho(v)]$.
- Now, if $x \in \text{St}_\rho(m)$, we have

$$1 \neq [\text{rst}_\rho(v)] \leq [\text{rst}_\sigma(u)]^x \land [\text{rst}_\sigma(u)] = [\text{rst}_\sigma(ux) \cap \text{rst}_\sigma(u)],$$

so $\text{rst}_\sigma(ux) = \text{rst}_\sigma(u)$.
- Hence $x \in \text{St}_\sigma(u)$.
- To finish, use transitivity of $G$ on all levels of $T_\rho$ and $T_\sigma$ to get
- $x \in \bigcap_{g \in G} \text{St}_\sigma(ug) = \text{St}_\sigma(n)$.
Two ways in which groups acting on trees can “look” like trees:
Two ways in which groups acting on trees can “look” like trees:

- Self-similarity/replication
  - strong replication in some examples: Gupta–Sidki 3-group ($p > 3$?), Grigorchuk group

Applications to commensurability and congruence subgroup problem.
Two ways in which groups acting on trees can “look” like trees:

- Self-similarity/replication
  - strong replication in some examples: Gupta–Sidki 3-group ($p > 3$?), Grigorchuk group
- subgroup structure of branch groups “detects” all trees on which group acts as branch group
  - Applications to commensurability and congruence subgroup problem.
Two ways in which groups acting on trees can “look” like trees:

- Self-similarity/replication
  - strong replication in some examples: Gupta–Sidki 3-group ($p > 3$?), Grigorchuk group
- subgroup structure of branch groups “detects” all trees on which group acts as branch group
  - Applications to commensurability and congruence subgroup problem.

Q How many “different” branch actions can a given group have? On what trees?
Thank you for your attention :)