Knapsack problems in products of groups

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Based on joint work with E.Frenkel and A.Ushakov

Non-commutative discrete optimization

Basic idea:

Take a classic algorithmic problem from computer science (traveling salesman, Post correspondence, knapsack,. . .) and translate it into group-theoretic setting.

Example: Post correspondence problem

Let A be an alphabet, $|A| \geq 2$.

The classic Post correspondence problem (PCP)

Given a finite set of pairs $(g_1, h_1), \ldots, (g_k, h_k)$ of elements of A^* determine if there is a non-empty word $w(x_1, \ldots, x_k) \in X^*$ such that $w(g_1, \ldots, g_k) = w(h_1, \ldots, h_k)$ in A^*.

Example: Post correspondence problem

Matching dominoes: top = bottom

g_{i_1}	g_{i_2}	g_{i_3}	$\dots$	g_{i_n}
h_{i_1}	h_{i_2}	h_{i_3}	$\dots$	h_{i_n}

Decidable if number of pairs is $k \leq 3$. Undecidable if $k \geq 7$.
Unknown if $4 \leq k \leq 6$.

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PCP in groups

Translating **PCP** to groups:

$A^* \rightsquigarrow$ f.g. group G,
words $g_i, h_i \rightsquigarrow$ group elements g_i, h_i given as words in generators,
word $w \rightsquigarrow$ group word,
right?

The above is trivial:
(a) $w = xx^{-1}$. Only allow non-trivial reduced words.
(b) G abelian, $w = [x, y]$. Only allow words that are not identities of G.

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Variations of **PCP** in groups turn out to be closely related to:

- double-endo-twisted conjugacy problem
  (find $w \in G$ s.t. $uw^{\varphi} = w^{\psi}v$),
- equalizer problem
  (find the subgroup of elements g s.t. $\varphi(g) = \psi(g)$),
- hereditary word problem
  (word problem in any quotient of G by a subgroup f.g. as a normal subgroup).

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The classic subset sum problem (SSP):

Given $a_1, \ldots, a_k, a \in \mathbb{Z}$ decide if

$$\varepsilon_1 a_1 + \ldots + \varepsilon_k a_k = a$$

for some $\varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\}$.

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Algorithmic set-up

Classic SSP is pseudopolynomial

- If input is given in unary, **SSP** is in **P**,
- if input is given in binary, **SSP** is **NP**-complete.

The complexity of **SSP**(G) does not depend on a finite generating set, but may depend on a generating set if infinite ones are allowed.

For example:

SSP$(\mathbb{Z})$

- **SSP**$(\mathbb{Z}) \in$ **P** if $\mathbb{Z}$ is generated by $\{1\}$,
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Group	Complexity	Why
Nilpotent	**P**	Poly growth
$\mathbb{Z} \wr \mathbb{Z}$	**NP**-complete	$\mathbb{Z}^\omega$, **ZOE**
Free metabelian	**NP**-complete	$\mathbb{Z} \wr \mathbb{Z}$
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$BS(m, n)$, $\|m\| \neq \|n\|$	**NP**-complete	Binary **SSP**($\mathbb{Z}$)
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Knapsack problems in groups

Three principle Knapsack type (decision) problems in groups:

SSP subset sum,

KP knapsack,

SMP submonoid membership.

The knapsack problem in groups

The classic knapsack problem (KP):

Given $a_1, \ldots, a_k, a \in \mathbb{Z}$ decide if

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for some non-negative integers $n_1, \ldots, n_k$.

The knapsack problem (KP) for G:

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Integer knapsack $=$ membership in product of cyclic groups.

The submonoid membership problem in groups

Submonoid membership problem (SMP):

Given a finite set $A = \{g_1, \ldots, g_k, g\}$ of elements of G decide if g belongs to the submonoid generated by A, i.e., if $g = g_{i_1}, \ldots, g_{i_s}$ for some $g_{i_j} \in A$.

If the set A is closed under inversion then we have the subgroup membership problem in G.

Bounded variations

It makes sense to consider the bounded versions of **KP** and **SMP**, they are always decidable in groups with decidable word problem.

The bounded knapsack problem (BKP) for G:

decide, when given $g_1, \ldots, g_k, g \in G$ and $1^m \in \mathbb{N}$, if $g =_G g_1^{\varepsilon_1} \ldots g_k^{\varepsilon_k}$ for some $\varepsilon_i \in \{0, 1, \ldots, m\}$.

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Bounded variations

Bounded submonoid membership problem (BSMP) for G:

Given $g_1, \ldots g_k, g \in G$ and $1^m \in \mathbb{N}$ (in unary) decide if g is equal in G to a product of the form $g = g_{i_1} \cdots g_{i_s}$, where $g_{i_1}, \ldots, g_{i_s} \in \{g_1, \ldots, g_k\}$ and $s \leq m$.

SSP and **BKP**:

- **NP**-complete in $\mathbb{Z} \wr \mathbb{Z}$, free metabelian, Thompson's F, $BS(m, n), m \neq \pm n$.
- **P**-time in f.g. v. nilpotent groups, hyperbolic groups, $BS(n, \pm n)$.

BSMP:

- **NP**-complete in $F_2 \times F_2$ (therefore **NP**-hard in any group that contains $F_2 \times F_2$, e.g. $B_{\geq 5}$, $GL(\geq 4, \mathbb{Z})$, partially commutative with induced $\square$.)
- **P**-time in f.g. v. nilpotent groups, hyperbolic groups.

Known results

KP:

- [MNU] **P**-time in abelian groups, hyperbolic groups.
- [Olshanski, Sapir, 2000] There is G with decidable **WP** and undecidable membership in cyclic subgroups.
- [Lohrey, 2013] Undecidable in $\mathrm{UT}_d(\mathbb{Z})$ if d is large enough.
- [Mischenko, Treyer, 2014] Undecidable in nilpotent groups of class ≥ 2 if $\gamma_c(G)$ is large enough. Decidable in $\mathrm{UT}_3(\mathbb{Z})$.

What about group-theoretic constructions?

Q1 Does **SSP** carry from G, H to $G * H$?

A1 That's not the right question.

Q2 Does **SSP** in $G \times H$ behave like the word problem or like the membership problem?

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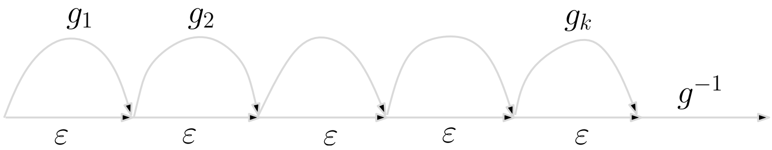

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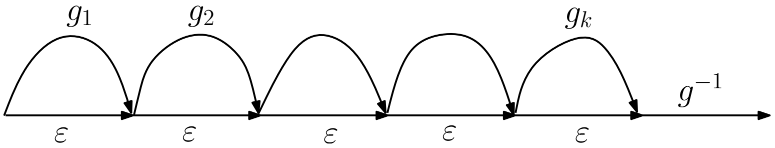
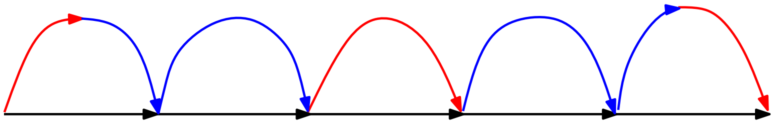


SSP and free products

Consider **SSP**$(G * H)$.

If some path reads trivial group element, then there is subpath in G or H that reads 1_G or 1_H, resp.



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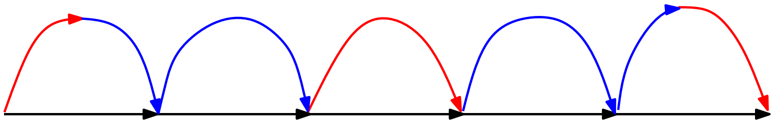

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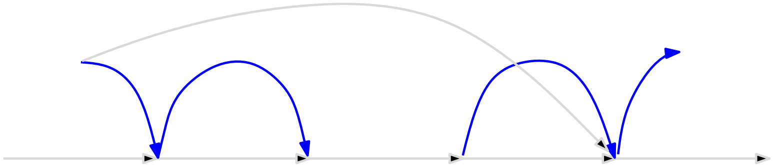

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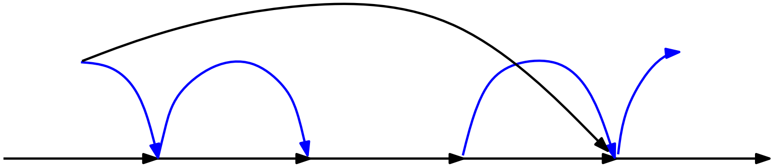

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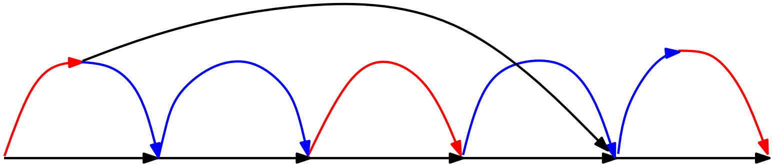

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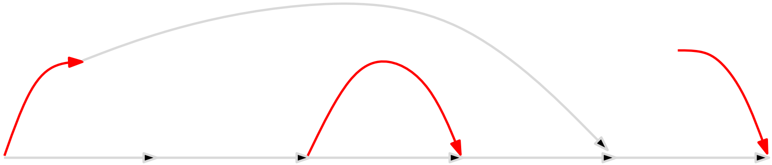

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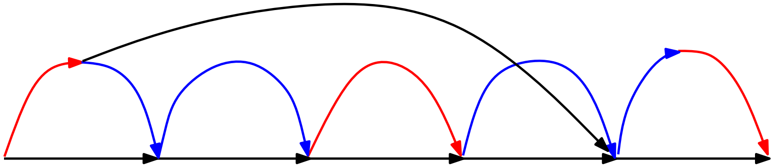

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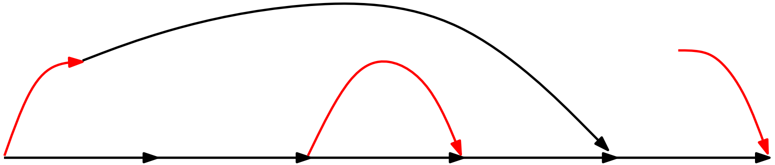

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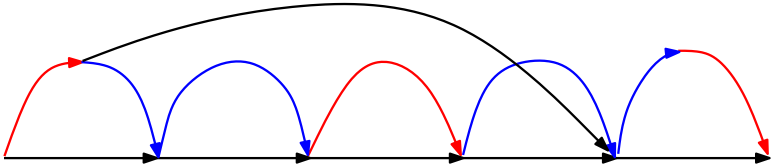

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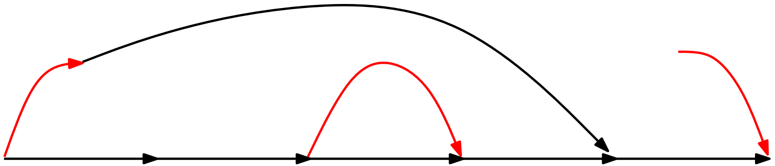

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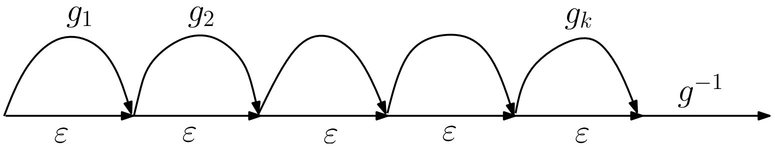
In this context, it is natural to consider so-called Acyclic Graph Problem:

The acyclic graph problem AGP(G, X)

Given an acyclic directed graph Γ labeled by letters in $X \cup X^{-1} \cup \{\varepsilon\}$ with two marked vertices, α and ω, decide whether there is an oriented path in Γ from α to ω labeled by a word w such that $w = 1$ in G.

AGP(G)

AGP(G) generalizes **SSP**(G) (i.e. **SSP**(G) is **P**-time reducible to **AGP**(G)):



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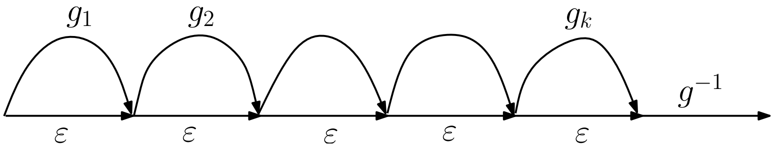

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Does **AGP**(G) reduce to **SSP**(G)?

We don't know. But in all G with **P**-time **SSP**(G) that we know, **AGP**(G) is also **P**-time, by essentially the same arguments:

- **AGP**(virtually f.g. nilpotent) $\in$ **P** by polynomial growth,
- **AGP**(hyperbolic) $\in$ **P** by logarithmic depth of Van Kampen diagrams.

Also, we know that **AGP**(G) **P**-time reduces to:

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Also, we know that **AGP**(G) **P**-time reduces to:

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AGP plays nicely with free products:

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Let G, H be finitely generated groups. Then **AGP**$(G * H)$ is **P**-time Cook reducible to **AGP**(G), **AGP**(H).

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If G, H are finitely generated groups such that $\mathbf{AGP}(G)$, $\mathbf{AGP}(H) \in \mathbf{P}$ then $\mathbf{AGP}(G * H) \in \mathbf{P}$.

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SSP, **BKP**, **BSMP**, **AGP** are polynomial time decidable in free products of finitely generated virtually nilpotent and hyperbolic groups in any finite number.

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KP and free products

What about Knapsack Problem **KP**$(G * H)$?

Difficulty: put a bound on exponents n_i in

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We can do it in

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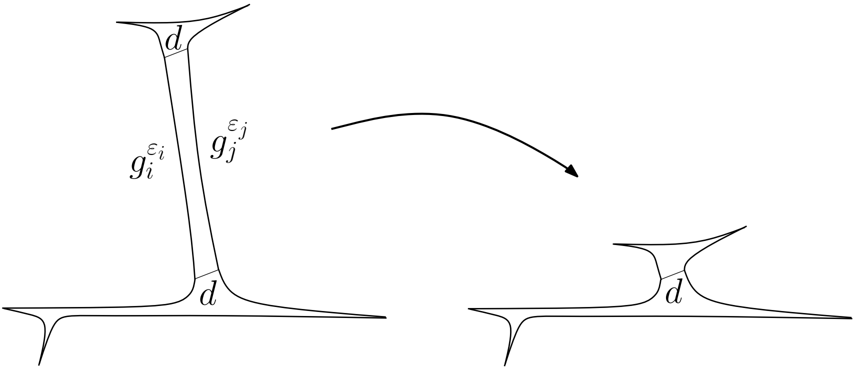

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Similar argument works in free products, which gives

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If G, H are groups such that $\mathbf{KP}(G), \mathbf{KP}(H) \in \mathbf{P}$, then $\mathbf{KP}(G * H)$ is $\mathbf{P}$-time reducible to $\mathbf{BKP}(G * H)$.

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$\mathbf{AGP}(G \times H)$ is decidable whenever $\mathbf{WP}(G), \mathbf{WP}(H)$ are decidable. What about complexity?

$\mathbf{AGP}(F_2 \times F_2)$ is **NP**-complete since $\mathbf{BSMP}(F_2 \times F_2)$ is, by a variation of Mikhailova construction.

By itself, this does not mean $\mathbf{SSP}(F_2 \times F_2)$ is **NP**-complete because we don't know whether $\mathbf{AGP}(G)$ reduces to $\mathbf{SSP}(G)$.

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Is $\mathbf{SSP}(F_2 \times F_2)$ **NP**-complete?

Answer: we don't know... but we know about $\mathbf{SSP}(F_2 \times F_2 \times \mathbb{Z})$!

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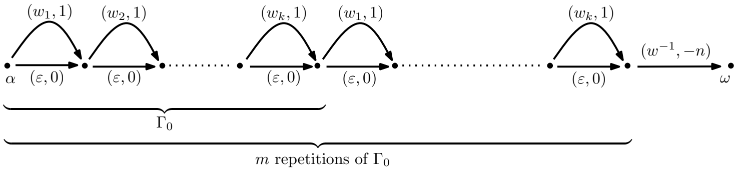

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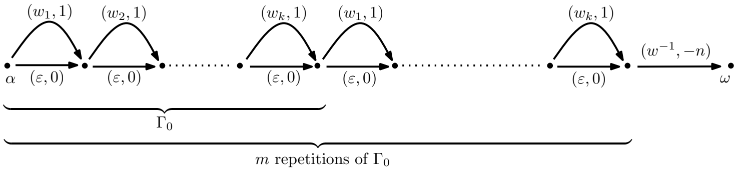

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