Knapsack problems in products of groups

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(Stevens Institute of Technology)

Group Theory Webinar, November 13, 2014

Based on joint work with E.Frenkel and A.Ushakov
Basic idea:

Take a classic algorithmic problem from computer science (traveling salesman, Post correspondence, knapsack, ...) and translate it into group-theoretic setting.
Let $A$ be an alphabet, $|A| \geq 2$.

The classic Post correspondence problem (PCP)

Given a finite set of pairs $(g_1, h_1), \ldots, (g_k, h_k)$ of elements of $A^*$ determine if there is a non-empty word $w(x_1, \ldots, x_k) \in X^*$ such that $w(g_1, \ldots, g_k) = w(h_1, \ldots, h_k)$ in $A^*$. 
Example: Post correspondence problem

Matching dominoes: top = bottom

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Translating **PCP** to groups:

$$A^* \Rightarrow \text{f.g. group } G,$$
words $$g_i, h_i \Rightarrow \text{group elements } g_i, h_i \text{ given as words in generators},$$
word $$w \Rightarrow \text{group word},$$
right?

The above is trivial:
(a) $$w = xx^{-1}$$. Only allow non-trivial reduced words.
(b) $$G$$ abelian, $$w = [x, y]$$. Only allow words that are not identities of $$G$$. 
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Variations of **PCP** in groups turn out to be closely related to:

- double-endo-twisted conjugacy problem
  
  \( (\text{find } w \in G \text{ s.t. } uw\varphi = w\psi v) \),

- equalizer problem
  
  \( (\text{find the subgroup of elements } g \text{ s.t. } \varphi(g) = \psi(g)) \),

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Non-commutative discrete optimization

The classic subset sum problem (SSP):

Given $a_1, \ldots, a_k, a \in \mathbb{Z}$ decide if

$$\varepsilon_1 a_1 + \ldots + \varepsilon_k a_k = a$$

for some $\varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\}$.

SSP for a group $G$:

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Algorithmic set-up

**Classic SSP is pseudopolynomial**

- If input is given in unary, SSP is in P,
- if input is given in binary, SSP is NP-complete.

The complexity of SSP(G) does not depend on a finite generating set, but may depend on a generating set if infinite ones are allowed.

For example:

**SSP(ℤ)**

- SSP(ℤ) ∈ P if ℤ is generated by \{1\},
- SSP(ℤ) is NP-complete if ℤ is generated by \{2^n | n ∈ ℕ\}.
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Knapsack problems in groups

Three principle Knapsack type (decision) problems in groups:

- **SSP** subset sum,
- **KP** knapsack,
- **SMP** submonoid membership.
The classic knapsack problem (**KP**):

Given $a_1, \ldots, a_k, a \in \mathbb{Z}$ decide if

$$n_1 a_1 + \ldots + n_k a_k = a$$

for some non-negative integers $n_1, \ldots, n_k$.

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There are minor variations of this problem, for instance, integer **KP**, when $n_i$ are arbitrary integers. They are all similar, we omit them here. The subset sum problem sometimes is called 0−1 knapsack.
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The submonoid membership problem in groups

Submonoid membership problem (SMP):

Given a finite set $A = \{g_1, \ldots, g_k, g\}$ of elements of $G$ decide if $g$ belongs to the submonoid generated by $A$, i.e., if $g = g_{i_1} \cdots g_{i_s}$ for some $g_{i_j} \in A$.

If the set $A$ is closed under inversion then we have the subgroup membership problem in $G$. 
It makes sense to consider the bounded versions of KP and SMP, they are always decidable in groups with decidable word problem.

The bounded knapsack problem (BKP) for $G$

decide, when given $g_1, \ldots, g_k, g \in G$ and $1^m \in \mathbb{N}$, if $g = g_1^{\varepsilon_1} \cdots g_k^{\varepsilon_k}$ for some $\varepsilon_i \in \{0, 1, \ldots, m\}$.

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Bounded variations

Bounded submonoid membership problem (BSMP) for $G$:

Given $g_1, \ldots, g_k, g \in G$ and $1^m \in \mathbb{N}$ (in unary) decide if $g$ is equal in $G$ to a product of the form $g = g_{i_1} \cdots g_{i_s}$, where $g_{i_1}, \ldots, g_{i_s} \in \{g_1, \ldots, g_k\}$ and $s \leq m$. 

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**Known results [MNU]**

**SSP and BKP:**
- **NP**-complete in $\mathbb{Z} \wr \mathbb{Z}$, free metabelian, Thompson’s $F$, $BS(m, n), m \neq \pm n$.
- **P**-time in f.g. v. nilpotent groups, hyperbolic groups, $BS(n, \pm n)$.

**BSMP:**
- **NP**-complete in $F_2 \times F_2$ (therefore **NP**-hard in any group that contains $F_2 \times F_2$, e.g. $B_{\geq 5}$, $GL(\geq 4, \mathbb{Z})$, partially commutative with induced $\Box$.)
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Known results

**KP:**

- [MNU] \( \mathsf{P} \)-time in abelian groups, hyperbolic groups.
- [Olshanski, Sapir, 2000] There is \( G \) with decidable \( \mathsf{WP} \) and undecidable membership in cyclic subgroups.
- [Lohrey, 2013] Undecidable in \( \mathsf{UT}_d(\mathbb{Z}) \) if \( d \) is large enough.
- [Mischenko, Treyer, 2014] Undecidable in nilpotent groups of class \( \geq 2 \) if \( \gamma_c(G) \) is large enough. Decidable in \( \mathsf{UT}_3(\mathbb{Z}) \).
What about group-theoretic constructions?

**Q1** Does **SSP** carry from $G, H$ to $G \ast H$?

**A1** That’s not the right question.

**Q2** Does **SSP** in $G \times H$ behave like the word problem or like the membership problem?

**A2** Both!
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What an instance of $\text{SSP}(G)$ looks like?

$g_1 \quad g_2 \quad g_k \quad g^{-1}$

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SSP and free products

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If some path reads trivial group element, then there is subpath in $G$ or $H$ that reads $1_G$ or $1_H$, resp.

Try to solve it using $\text{SSP}(G)$ and $\text{SSP}(H)$. 
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In this context, it is natural to consider so-called Acyclic Graph Problem:

**The acyclic graph problem $\text{AGP}(G, X)$**

Given an acyclic directed graph $\Gamma$ labeled by letters in $X \cup X^{-1} \cup \{\varepsilon\}$ with two marked vertices, $\alpha$ and $\omega$, decide whether there is an oriented path in $\Gamma$ from $\alpha$ to $\omega$ labeled by a word $w$ such that $w = 1$ in $G$. 

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Does \( \text{AGP}(G) \) reduce to \( \text{SSP}(G) \)?

We don’t know. But in all \( G \) with \( \text{P}-\text{time} \ \text{SSP}(G) \) that we know, \( \text{AGP}(G) \) is also \( \text{P}-\text{time} \), by essentially the same arguments:

- \( \text{AGP}(\text{virtually f.g. nilpotent}) \in \text{P} \) by polynomial growth,
- \( \text{AGP}(\text{hyperbolic}) \in \text{P} \) by logarithmic depth of Van Kampen diagrams.

Also, we know that \( \text{AGP}(G) \ \text{P}-\text{time} \) reduces to:

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- $\text{SSP}(G \ast F_2)$. 

**Question**

Does $AGP(G)$ reduce to $SSP(G)$?

We don’t know. But in all $G$ with $P$-time $SSP(G)$ that we know, $AGP(G)$ is also $P$-time, by essentially the same arguments:

- $AGP($virtually f.g. nilpotent$) \in P$ by polynomial growth,
- $AGP($hyperbolic$) \in P$ by logarithmic depth of Van Kampen diagrams.

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AGP plays nicely with free products:

**Theorem**

Let $G, H$ be finitely generated groups. Then $\text{AGP}(G \ast H)$ is $\text{P}$-time Cook reducible to $\text{AGP}(G), \text{AGP}(H)$.

Proof: same as what we tried to do with $\text{SSP}$, only this time it works.
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If $G, H$ are finitely generated groups such that $\text{AGP}(G), \text{AGP}(H) \in P$ then $\text{AGP}(G \ast H) \in P$.

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$\text{SSP}$, $\text{BKP}$, $\text{BSMP}$, $\text{AGP}$ are polynomial time decidable in free products of finitely generated virtually nilpotent and hyperbolic groups in any finite number.
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Similar argument works in free products, which gives

**Theorem**

If $G, H$ are groups such that $KP(G), KP(H) \in P$, then $KP(G \ast H)$ is $P$-time reducible to $BKP(G \ast H)$.

**Corollary**

If $G, H$ are groups such that $AGP(G), AGP(H) \in P$ and $KP(G), KP(H) \in P$ then $KP(G \ast H) \in P$.

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KP and free products

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\( \text{AGP}(F_2 \times F_2) \) is \( \text{NP} \)-complete since \( \text{BSMP}(F_2 \times F_2) \) is, by a variation of Mikhailova construction.

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Is \( \text{SSP}(F_2 \times F_2) \) \( \text{NP} \)-complete?

**Answer:** we don’t know... but we know about \( \text{SSP}(F_2 \times F_2 \times \mathbb{Z})! \)
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BSMP\((G)\) vs SSP\((G \times \mathbb{Z})\):  

BSMP\((G)\) reduces to SSP\((G \times \mathbb{Z})\):

\[
(w_1, 1) \to (\varepsilon, 0) \\
(w_2, 1) \to (\varepsilon, 0) \\
(w_k, 1) \to (\varepsilon, 0) \\
(w_1, 1) \to (\varepsilon, 0) \\
(w_k, 1) \to (w^{-1}, -n) \\
\alpha \to \omega
\]

\Gamma_0

\(m\) repetitions of \(\Gamma_0\)

Corollary

SSP\((F_2 \times F_2 \times \mathbb{Z})\) is NP-complete.
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SSP and direct products

Observation: \( \text{AGP}(G) \) and \( \text{AGP}(G \times \mathbb{Z}) \) are \( \mathsf{P} \)-time equivalent.

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There are groups \( G, H \) such that \( \text{SSP}(G), \text{SSP}(H) \in \mathsf{P} \), but \( \text{SSP}(G \times H) \) is \( \mathsf{NP} \)-complete.

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Some of (many) open questions:

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