Linear groups with Borel's property

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This talk covers joint work with Michael Larsen

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Talk Outline

Some definitions

Armand Borel's Theorem on free groups

Extending Borel's Theorem to other groups

A bit about the proofs

Double word maps and an Open Question

Linear groups

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- A subgroup of $GL(n,\mathbb{C})$ is a *linear algebraic group* if it is defined by polynomial equations.

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is the prototypical example of a semisimple group.
- Here are a couple of non-examples: Connected solvable groups (i.e., unipotent upper triangular matrices over $\mathbb{C}$) and $GL(n,\mathbb{C})$).

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- We say a nontrivial element $a \in A$ is *detectable by G rel V*, if there exists a homomorphism $A \to G$ where the image of a is not contained in V.
- If any (nontorsion) element, $a \in A$, is detectable by G rel V for any proper subvariety V in G, then we say A is *(weakly) G-free*.

A. Borel's Theorem

Theorem (Armand Borel, 1983)

Let G be a semisimple linear algebraic group. Let F be a free group. Then F is G-free.

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- That is, the group F is not G-free.

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Because $GL(n, \mathbb{C})$ didn't work.The next most natural liner algebraic groups to look at are $SL(n, \mathbb{C})$, $SO(3)$, etc. Also, a lot is known about them, and there are connections to the Banach-Tarski Paradox.

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- We say a group A has *Borel's property* if it is weakly G-free for any semisimple group G. Let the class of groups with Borel's property be denoted by $\mathscr{L}$.
- Let the class of torsion-free groups with Borel's property be denoted by $\mathscr{B}$.
- Clearly, $\mathscr{B} \subset \mathscr{L}$.
- Borel's Theorem may be stated simply as: All free groups are in $\mathscr{L}$.

Residually free groups

- Recall that a group A is *residually free* if for any $a \in A$, there exists a homomorphism $\phi : G \to F$, F free, such that $\phi(a) \neq 1$.
- Let A be a residually free group. Let G be a semisimple group and V a proper subgroup of G. Let $a \in A$, then because A is residually free, there exists $\phi : A \to F$, such that $\phi(a) \neq 1$. Since F is G-free, there exists a homorphism $\psi : F \to G$ such that $\psi(\phi(a)) \neq 1$. Thus, A has Borel's property.

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- If A, B are residually free, then $A \times B$ is residually free.
- What about the last two properties?

Solvable groups in $\mathscr{B}$

We've shown that $\mathscr{B}$ satisfies a strong Tits' alternative:

Theorem (Khalid Bou-Rabee, Michael Larsen, 2014)

Let Γ be a finitely generated group that is in $\mathscr{B}$. Then Γ contains a free group or is a free abelian group.

Free products of groups in $\mathscr{B}$

Proposition (Khalid Bou-Rabee, Michael Larsen, 2014)

If $A = F_2 \times \mathbb{Z}$ *and* $B = \mathbb{Z}$. *Then* $A, B \in \mathscr{B}$, *but* $A * B$ *is not in* $\mathscr{B}$.

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Is $\mathscr{B}$ simply the group of residually free groups?

The answer

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No.

Groups with Borel's property that have torsion

By putting Borel's methods under a microsope, we were able to extend his result to other groups:

Theorem (Khalid Bou-Rabee, Michael Larsen, 2014)

Let p be a prime such that $p \equiv 1 \mod 3$. Then the group $\mathbb{Z}/p * \mathbb{Z}/p$ has Borel's property. That is, this group is in $\mathscr{L}$.

A group that is in $\mathscr{B}$ but not residually free.

Here is the first example we found. Let

$$K = \left\langle a_1, \ldots, a_7, b : ba_1b^{-1} = a_2, \ldots, ba_6b^{-1} = a_7, ba_7b^{-1} = a_1 \right\rangle,$$

then this group is in $\mathscr{B}$, but is not residually free.

Why is K not residually free?

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Residually free groups are residually finite 2-group. The action of b on the free group $\langle a_1, \ldots, a_7 \rangle$ has order two!

Why is K in $\mathscr{B}$?

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The group K is contained inside $\mathbb{Z}/7 * \mathbb{Z}/7$, which is in $\mathscr{L}$ from two slides ago.

What about surface groups?

- A (oriented) *surface group* is the fundamental group of a (oriented) two-dimensional manifold.
- By Baumslag's result, all oriented surface groups are in $\mathscr{B}$.
- What about nonoriented surfaces?

The Klein bottle

The Klein bottle is virtually abelian, and thus is neither free abelian nor contains a nonabelian free group. So by the Altternative for $\mathscr{B}$, the Klein bottle is not in $\mathscr{B}$.

Projected sums of k planes

- We are left with one relator groups of the form

$$\left\langle a, b, c, \ldots, z : a^2 b^2 \cdots z^2 = 1 \right\rangle.$$

- Baumslag showed that if the number of letters is greater than 3, then the group in question is residually free, so those have Borel's property.
- We are left with $\left\langle a, b : a^2 b^2 = 1 \right\rangle$ and $\left\langle a : a^2 = 1 \right\rangle$. The latter group is finite, and thus not in $\mathscr{B}$.
- The former group was shown to *not* be residually free by R. C. Lyndon and M. P. Schützenberger in 1962. They did this by studying the equation $a^M = b^N c^P$ in a free group.

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Projected sums of 3 planes

Theorem (Khalid Bou-Rabee and Michael Larsen, 2014)
The group $\langle a, b, c : a^2 b^2 c^2 = 1 \rangle$ *is in* $\mathscr{B}$.

We also handled other groups, including some other Fuchsian groups and von Dyck groups. See our joint paper 'Linear groups with Borel's property' (2014).

Extending Borel's proof

What is the main difficulty in extending Borel's proof to other groups? To understand this, let's look at a brief outline of the proof. Let F be a nonabelian free group.

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- Show that F is $SL(2, \mathbb{C})$-free. This follows from a quick dimension counting argument.
- Show that F is $SL(3, \mathbb{C})$-free. This is the main difficulty. Ping-pong lemma gives *many* images of F, but they all have a single eigenvalue 1. The trick that Borel uses is to find representations of F that factor through a division algebra, so *none* of the eigenvalues can be one!
- Now use the structure theory of semisimple algebraic groups, to conclude that F is G-free for all semisimple G.

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- Now use the structure theory of semisimple algebraic groups, to conclude that F is G-free for all semisimple G.

The free group is very nice, in that finding homomorphisms from F is very easy! For other groups, their representation varieties are no longer connected, so one has to be very careful with dimension counting arguments.

A theorem of Breuillard, Green, Guralnick, and Tao

Theorem (Breuillard, Green, Guralnick, and Tao, 2012)

Let w_1, w_2 be two elements in a free group of rank 2. Let a, b be generic elements of a semisimple Lie group G over an algebraically closed field. Then $w_1(a, b)$ and $w_2(a, b)$ generate a Zariski-dense subgroup of G.

Using this theorem, they were able to prove results on *expanding generators* (elements that generate a Cayley graph which is an expander).

Their question

Question (Breuillard, Green, Guralnick, and Tao, 2012)

Can one characterize the set of pairs of words (w_1, w_2) in the free group F_2 such that the double word map $G \times G \to G \times G$ given by

$$e_{w_1,w_2}(a,b) = (w_1(a,b), w_2(a,b))$$

is dominant?

Our answer

Theorem (Khalid Bou-Rabee and Michael Larsen)

Yes, if $w_1 = [x_1, x_2] \cdots [x_{2k-1}, x_{2k}]$ *where* $k \geq 2$*. Let* w_2 *be a word not in the normal closure of* w_1*. Then the double-word map defined by* w_1, w_2 *is dominant.*

Thank you!

The End.