Subgroup Conjugacy Separability for Groups

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Webinar “GT”, NY, 3.04.14

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4. Hurwitz' problem
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Part I.
New residual properties of groups

Known residual properties of groups:

- residually finite groups (RF)
- conjugacy separable groups (CS)
- locally extended residually finite groups (LERF)

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New residual property of groups (SCS-property):

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The following groups are SCS:

- virtually polycyclic groups (Grunewald and Segal)
- free groups and some virtually free groups (B. and Grunewald)
- orientable surface groups (B. and Bux)
- $A * B$ if A, B are SCS and LERF (B. and Elsawy)

Another useful property: SICS

For $A, B \leqslant G$, we say that A is conjugate into B if there exists $g \in G$ with $A^g \leqslant B$. We write

$$A \underset{G}{\rightsquigarrow} B$$

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Def. A group G is called subgroup into-conjugacy separable (SICS), if for any two f.g. subgroups $H_1, H_2 \leqslant G$ the following implication holds:

$$H_2 \underset{G}{\not\leadsto} H_1 \Longrightarrow \text{ there exists a fin. quotient } \overline{G} : \ \overline{H_2} \underset{\overline{G}}{\not\leadsto} \overline{H_1}$$

How to prove that G is SCS (a strategy):

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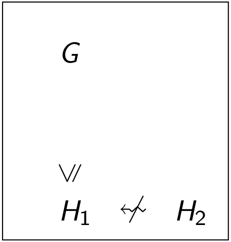
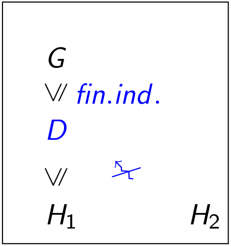
If



then $\exists D$:



3. Use coverings if G is "geometric".

For which groups SICS implies SCS?

Lem. Suppose that a group G does not contain a f.g. subgroup H and an element g such that $H < H^g$. Then for G we have (SICS $\Longrightarrow$ SCS).

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Cor. For free and surface groups, we have (SICS $\Longrightarrow$ SCS).

Proof (for free groups). If the assumption of Lemma is not satisfied, then

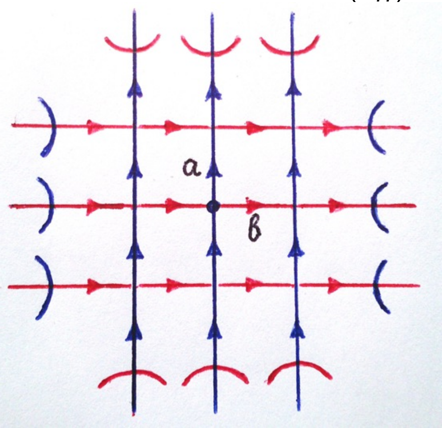
$$H < H^g < H^{g^2} < \dots$$

This contradicts Takahasi's result that for any strictly ascending infinite chain of f.g. subgroups in a free group the ranks of the members of this chain are unbounded.

Part II.
Proof that free groups are SICS

Proof that F is SICS

Step 0. (Notations) We realize $F = \pi_1(R)$. For each $H \leqslant F$, there is a covering $\Gamma_H \to R$ with $\pi_1(\Gamma_H) = H$. There are edges in Γ_H, which we call entries in and exists from $Core(\Gamma_H)$.



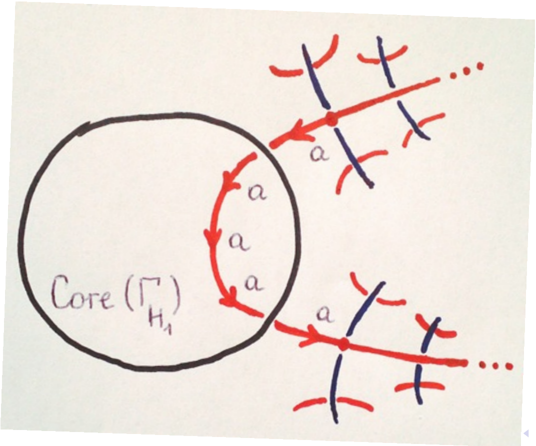
$$H := \langle [a, b], [a, b^{-1}], [a^{-1}, b], [a^{-1}, b^{-1}] \rangle \leqslant F(a, b)$$

Proof that F is SICS

Remember, that we have $H_1, H_2 \leqslant F$ such that $H_2 \not\hookrightarrow H_1$.

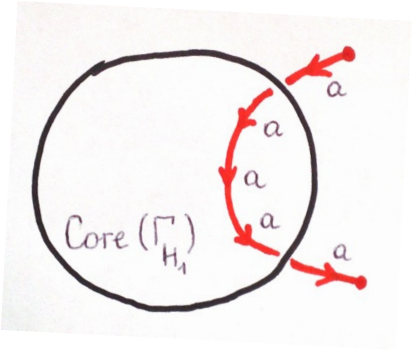
Step 1. (M.Hall theorem for H_1) We construct a subgroup $D \leqslant F$ of finite index in F which contains H_1 as a free factor.

1.1. Take the covering space Γ_{H_1}.



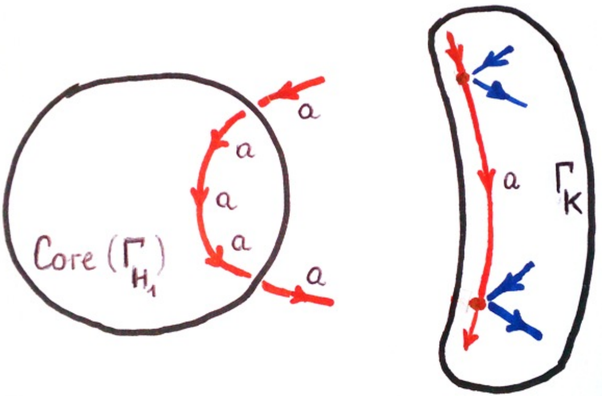
Proof that F is SICS

1.2. Cut out infinite trees as it shown below.



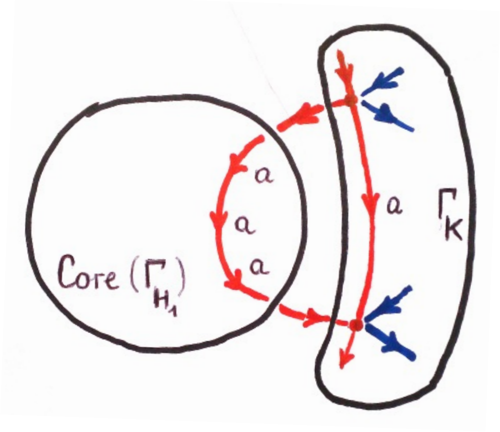
Proof that F is SICS

1.3. Take an arbitrary finite covering $\Gamma_K \to R$.



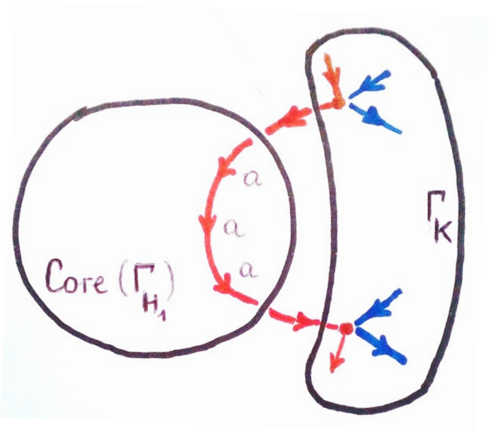
Proof that F is SICS

1.4. Glue these two pieces.



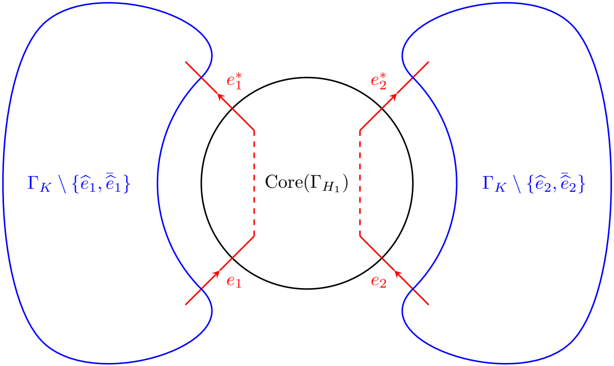
Proof that F is SICS

1.5. Delete the edge with the label a.



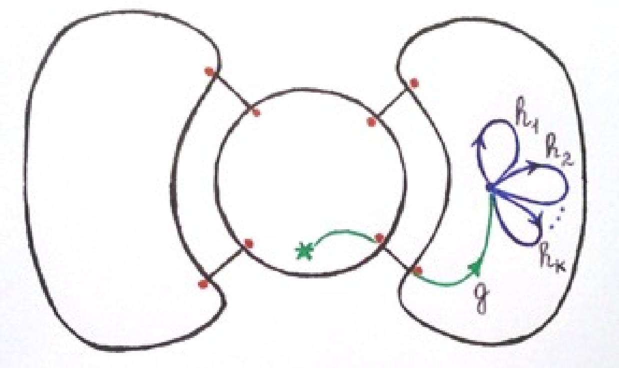
Proof that F is SICS

We get a finite covering Γ_D containing $Core(\Gamma_{H_1})$.
Thus, D is a subgroup of finite index in F containing H_1 as a free factor.



Proof that F is SICS

Step 2. (Put H_2 in the play) Let $H_2 = \langle h_1, h_2, \ldots, h_n \rangle$. We don't want the situation that is shown below, since it would mean that $H_2^g \leqslant D$.



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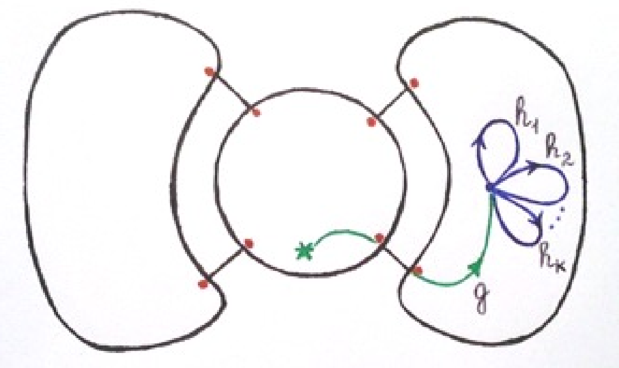


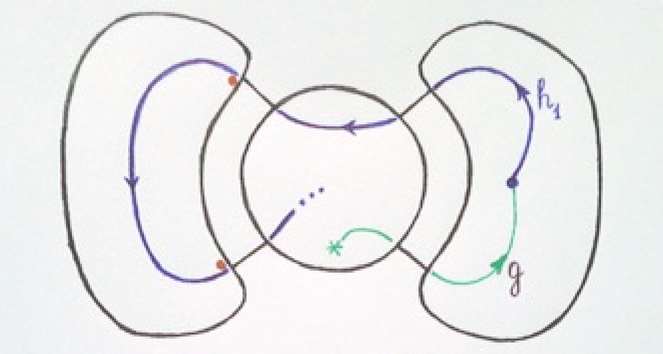
To avoid this situation, we take Γ_K that does not contain small loops, i.e., nontrivial loops of length up to $C := max\{|h_i|\} + 1$.

Proof that F is SICS

Step 3. (End of the proof) We prove that if we take Γ_K without loops of length up to C, then $H_2 = \langle h_1, \ldots, h_n \rangle$ is not conjugate into D.

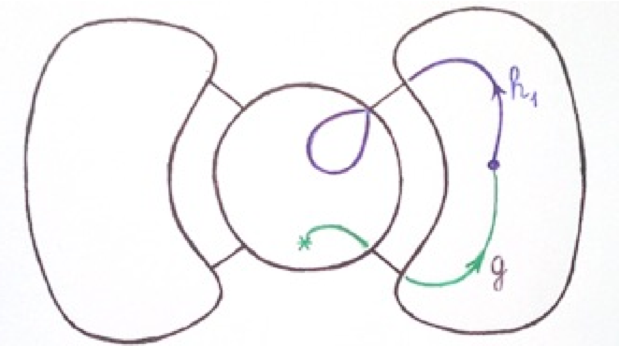
Suppose the contrary: $H_2^g \leqslant D$. Where the loops with labels h_i can lie?

Case 1. This cannot happen.



Proof that F is SICS

Hence we have



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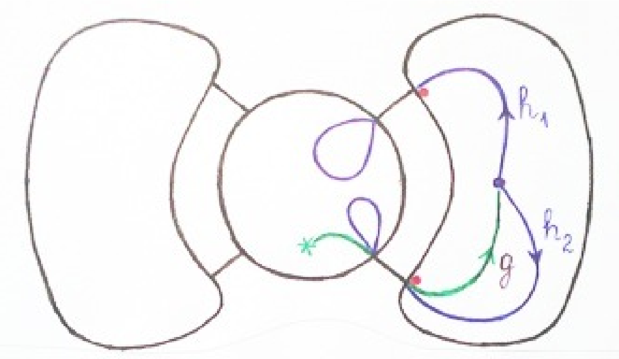

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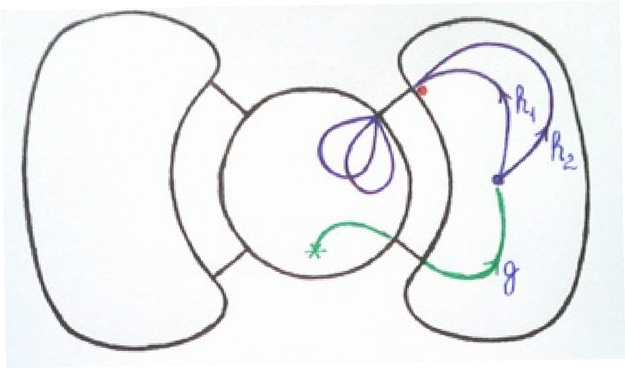
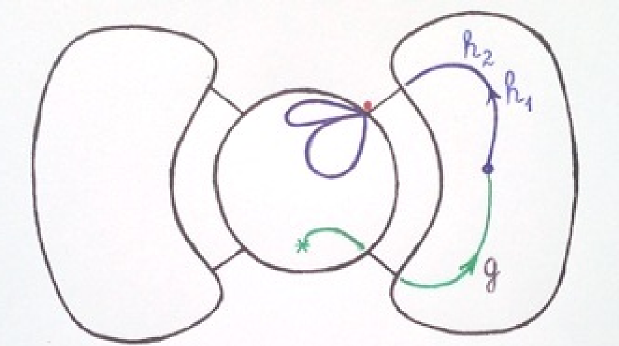


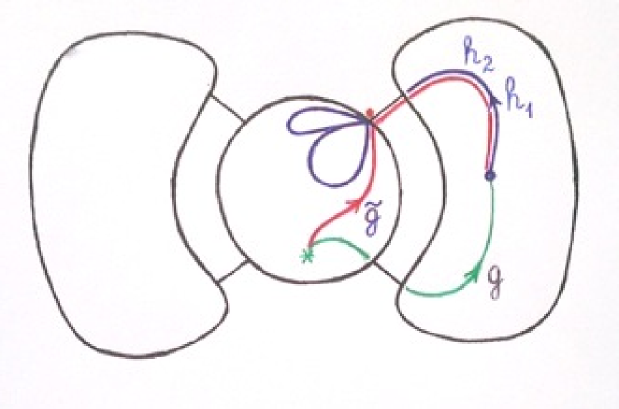
Proof that F is SICS

Case 4. This can happen.



Proof that F is SICS

Case 4. This can happen. But in this case $H_2^{\widetilde{g}} \leqslant H_1$ that contradicts our assumption. Thus, H_2 is not conjugate into D.



Part III.
Proof that surface groups are SICS

Visualization of subgroups

Let G be a group. For any fin.gen. subgroup $H \leqslant G$, we choose

$$H \leqslant H^* \overset{fin.ind}{\leqslant} G.$$

Useful: If G is a "geometric group", then, given $H \leqslant G$, we will choose H^* so that H is "geometrically embedded" in H^*.

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Ex:
1) M. Hall Thm. For any fin. gen. subgroup H of a free group G, there exists a finite index subgroup H^* in G such that H is a free factor of H^*.
2) P. Scott Thm. A similar, but not the same statement for surface groups.

Recall the Definition of SICS

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Def. Suppose that for any fin.gen. $H_1 \leqslant G$ and for any fin.gen. $H_2 \leqslant G$ we have

$$H_2 \underset{G}{\not\rightsquigarrow} H_1 \Longrightarrow \text{ there exists a fin. quotient } \overline{G} : \ \overline{H_2} \underset{\overline{G}}{\not\rightsquigarrow} \overline{H_1}.$$

Then G is called SICS.

Star Lemma for SICS

Let G be a group. For any fin.gen. subgroup $H \leqslant G$, we choose

$$H \leqslant H^* \overset{fin.ind}{\leqslant} G.$$

Lem. Suppose that for any fin.gen. $H_1 \leqslant G$ and for any fin.gen. $H_2 \leqslant H_1^*$ we have

$$H_2 \underset{H_1^*}{\not\rightsquigarrow} H_1 \Longrightarrow \text{ there exists a fin. quotient } \overline{H_1^*}: \ \overline{H_2} \underset{\overline{H_1^*}}{\not\rightsquigarrow} \overline{H_1}.$$

Then G is SICS.

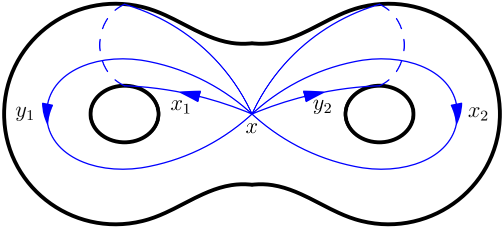
Scott' Theorem

Thm. (P. Scott) Let S be a closed surface. For any fin. gen. subgroup $H \leqslant \pi_1(S, x)$, there exists a finitely sheeted covering map $p : (\widetilde{S}, \widetilde{x}) \to (S, x)$ such that H is realized by a subsurface in $\widetilde{S}$.

The latter means that there exists an incompressible compact subsurface $A \subseteq \widetilde{S}$ containing $\widetilde{x}$ such that $p_*(\pi_1(A, \widetilde{x})) = H$.

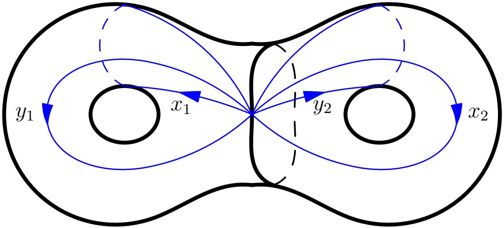
Thus, we can set $H^* := p_*(\pi_1(\widetilde{S}, \widetilde{x}))$.

Example 1 to Scott' Theorem

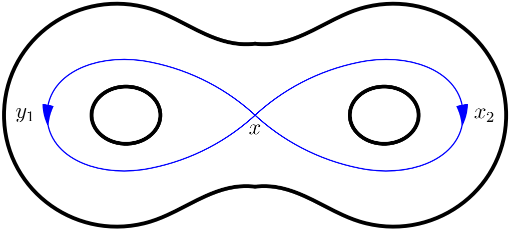


$\pi_1(S, x) = \langle x_1, x_2, y_1, y_2 \mid [x_1, y_1][x_2, y_2] = 1 \rangle$, where $[a, b] := a^{-1}b^{-1}ab$

The subgroup $\langle x_1, y_1 \rangle$ can be realized by a subsurface in S.



Example 2 to Scott' Theorem



The cyclic subgroup $\langle y_1 x_2^{-1} \rangle \leqslant \pi_1(S, x)$ cannot be realized by a subsurface in S.

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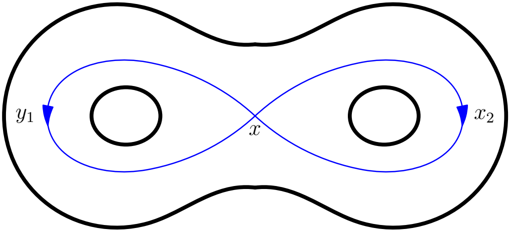

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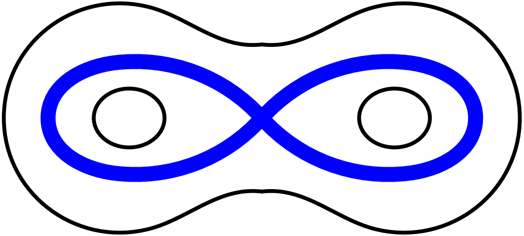
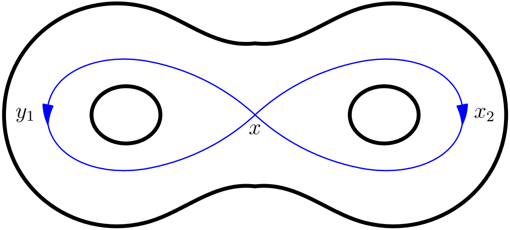


Example 2 to Scott' Theorem (continuation)

How to construct a covering $\widetilde{S} \to S$ such that the subgroup $\langle y_1 x_2^{-1} \rangle \leqslant \pi_1(S, x)$ is realized by a subsurface in $\widetilde{S}$?



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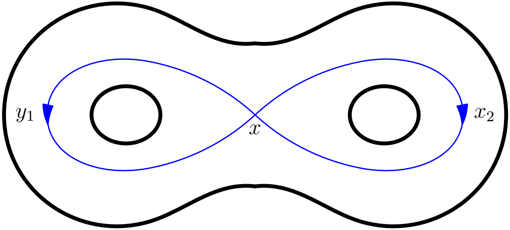

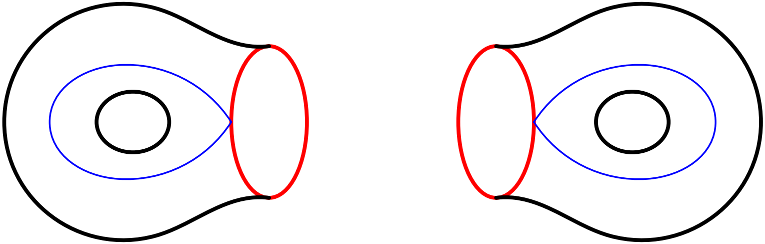

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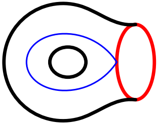

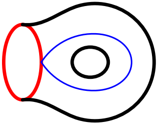

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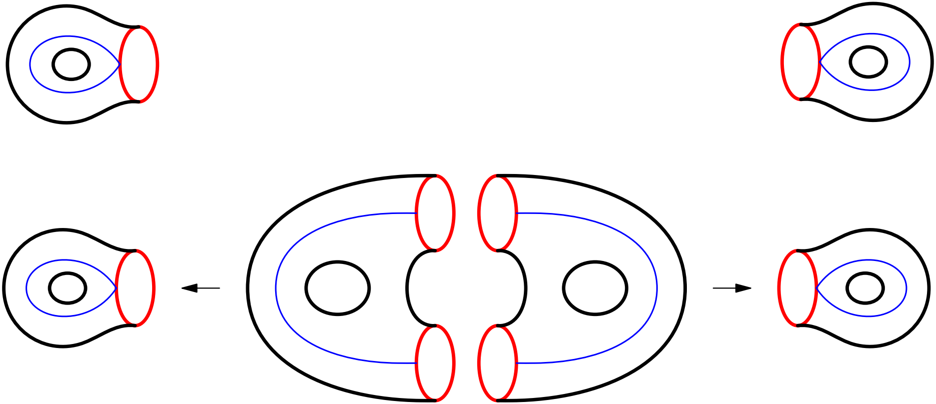

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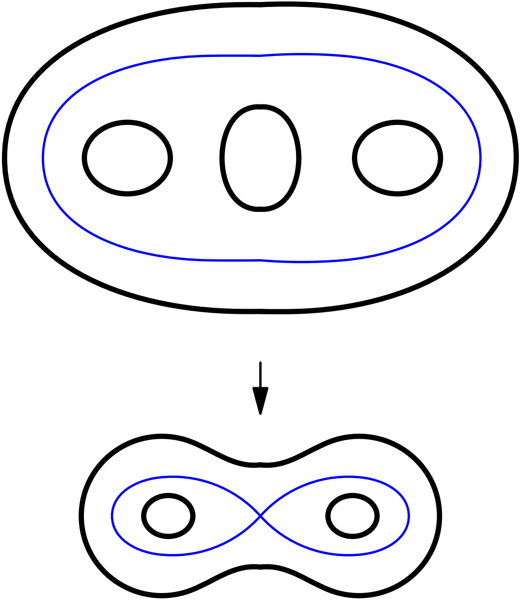

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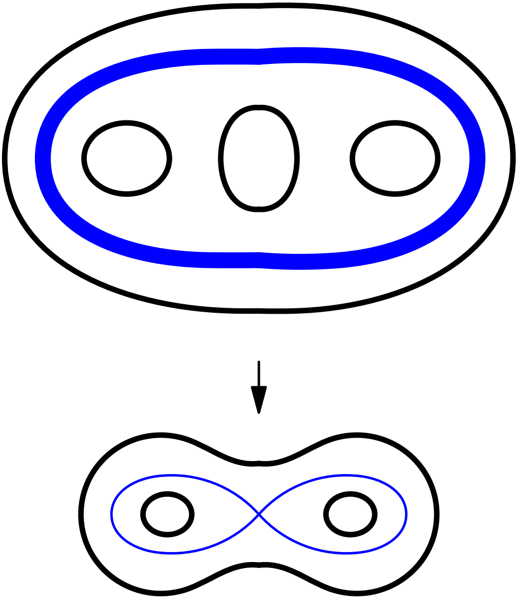
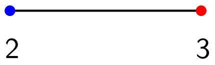


An improvement of Scott' Theorem

P.Scott Theorem (improved). Let S be a closed surface with $\chi(S) < 0$. For any finitely generated subgroup $H \leqslant \pi_1(S, x)$, there exists a finitely sheeted covering map $p : (\widetilde{S}, \widetilde{x}) \to (S, x)$ such that

1) H is **realized** in $\widetilde{S}$, i.e., there exists an incompressible compact subsurface $A \subseteq \widetilde{S}$ containing $\widetilde{x}$ with $p_*(\pi_1(A, \widetilde{x})) = H$;
2) A has a **good shape**, i.e., $B := S \setminus A$ is a connected surface and $genus(B) > 0$.

Branched coverings of graphs (Example 1)



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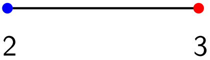

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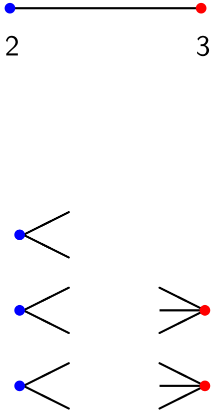

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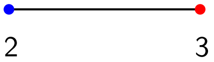
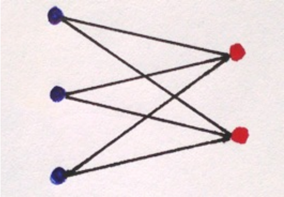
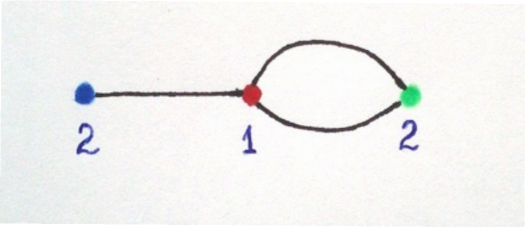




Branched coverings of graphs (Example 2)



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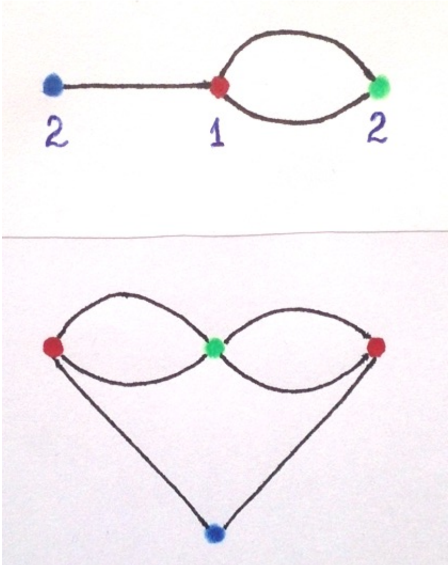
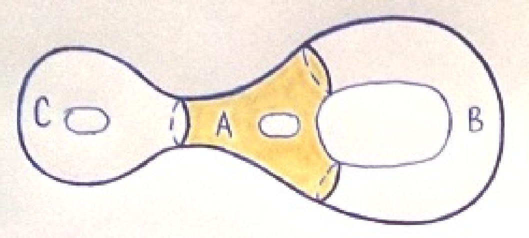


Creating a genus in all components of $S \setminus A$



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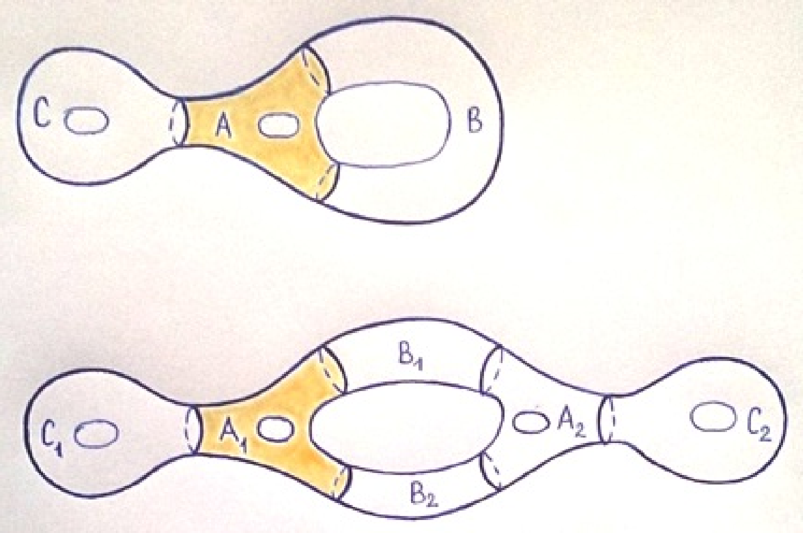
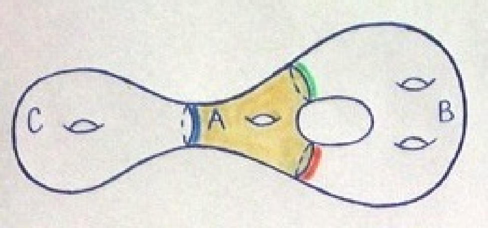


Creating a unique complementary component to A



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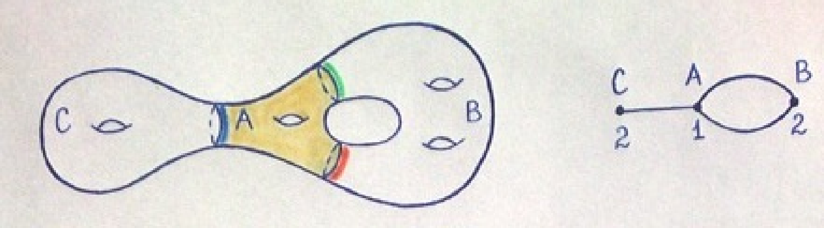

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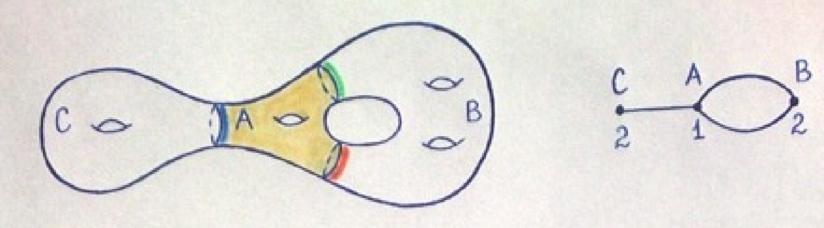

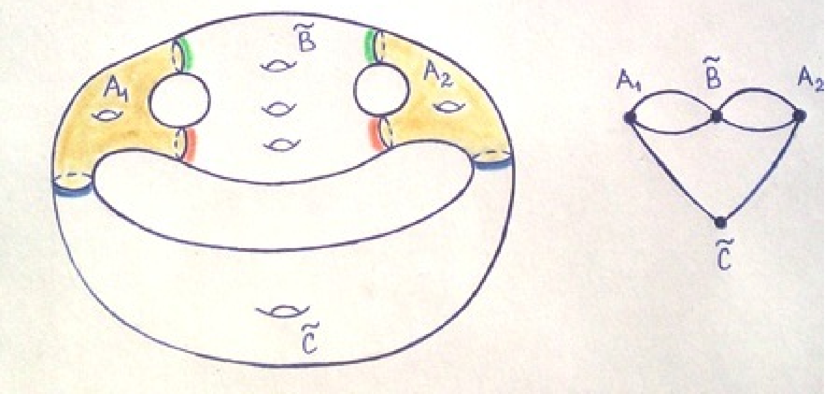

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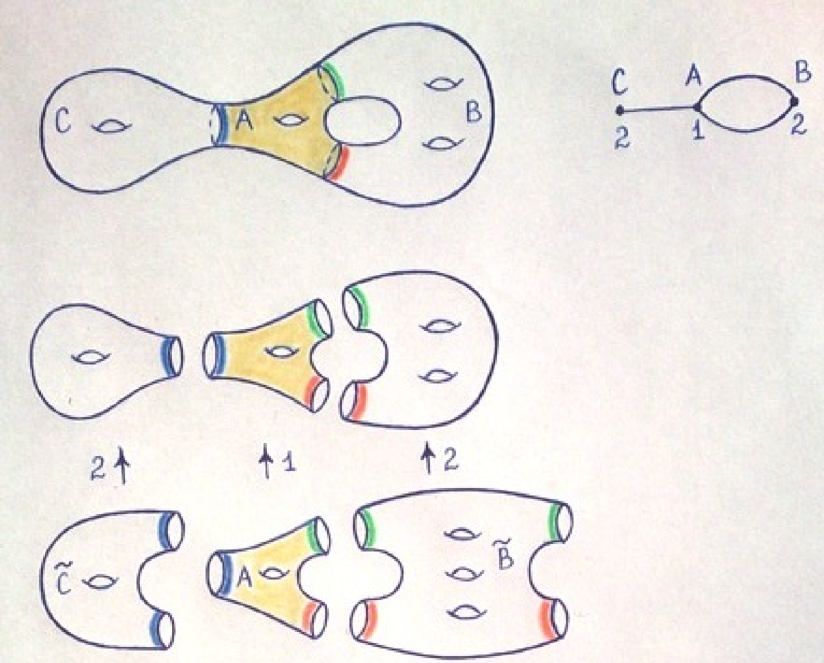
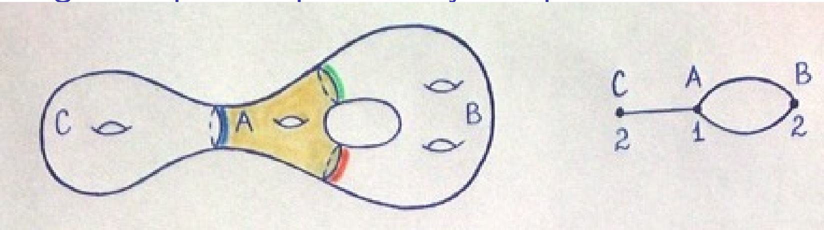
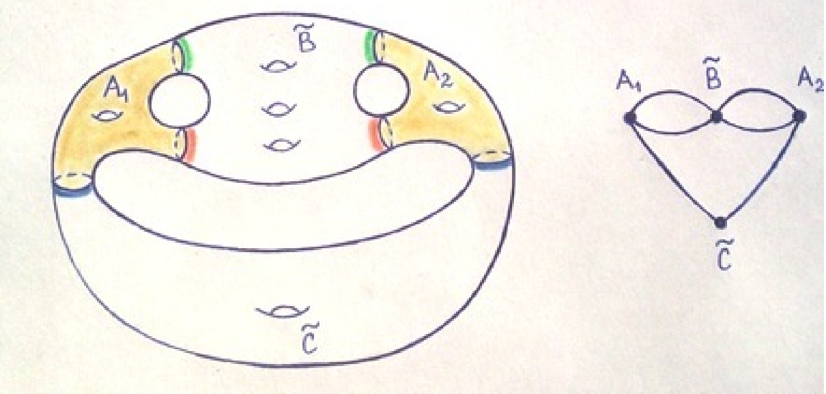
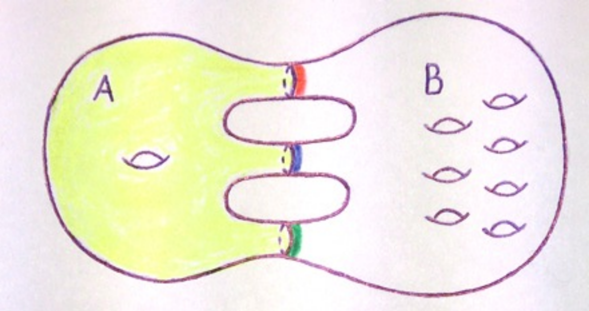


Creating a unique complementary component to A





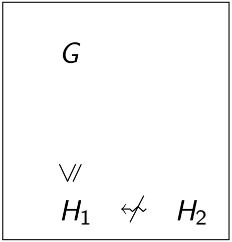
Proof that surface groups are SICS (a preparation)



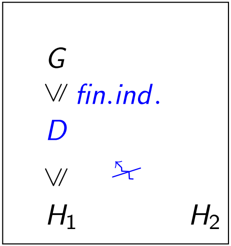
We want to prove that $G := \pi_1(S)$ is SICS. Let $H_1, H_2 \leqslant \pi_1(S)$ be fin. gen. and such that $H_2 \not\rightsquigarrow H_1$. Using the Star Lemma and the improvement of Scott' Theorem, we may assume the following:

Assumption. $H_1 \leqslant \pi_1(S)$ is realized by an incompressible subsurface $A \subset S$ s.t. $B := S \setminus A$ is a connected surface with $genus(B) > 0$.

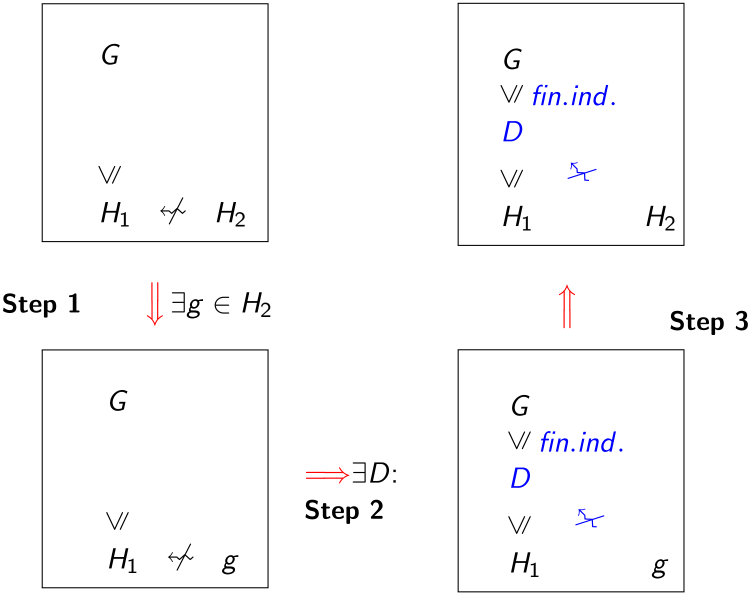
What we want to prove for $G = \pi_1(S)$



$\Longrightarrow \exists D$:

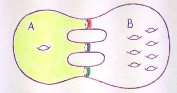


Proof in three steps



Step 1

Claim. Let $H_1, H_2 \leqslant \pi_1(S)$ be fin.gen. Suppose that H_1 is realized by a surface $A \subset S$ such that $B := S \setminus A$ is a connected surface.



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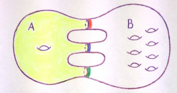

If each element $g \in H_2$ is conjugate into H_1, then the whole H_2 is conjugate into H_1.

Proof. We may assume that H_2 is noncyclic.

- The decomposition $S = A \cup B$ induces a graph of groups decomposition of $\pi_1(S)$ with two vertex groups $\pi_1(A)$ and $\pi_1(B)$, and with cyclic edge groups corresponding to the common boundary components $R_1, \dots, R_n$ of A and B.

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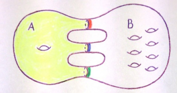

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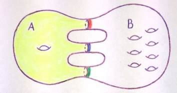

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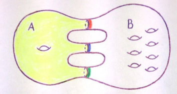
- $G = \pi_1(S)$ acts on the Bass-Serre tree T with vertex stabilizers conjugate to $\pi_1(A)$ and to $\pi_1(B)$.
- Each $g \in H_2$ acts elliptically on T (since $g \rightsquigarrow H_1 = \pi_1(A)$). Hence H_2 has a global fixed vertex in T.

Step 1 (continuation)

Claim. Let $H_1, H_2 \leqslant \pi_1(S)$ be fin.gen. Suppose that H_1 is realized by a surface $A \subset S$ such that $B := S \setminus A$ is a connected surface.



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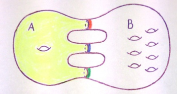

If each element $g \in H_2$ is conjugate into H_1, then the whole H_2 is conjugate into H_1.

Continuation of the proof.

- If H_2 fixes a vertex of type A, then H_2 is conjugate of $\pi_1(A)$ and we are done. Suppose that H_2 fixes a vertex of type B. Recall that each $g \in H_2$ fixes a vertex of type A.

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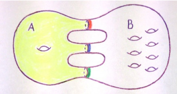

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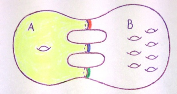

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- Thus, each $g \in H_2$ is conjugate to a power of some $a \in \{a_1, \dots, a_n\}$.

Step 1 (continuation)

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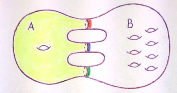

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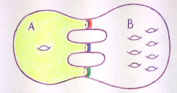

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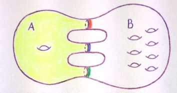

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Continuation of the proof.

- Each $g \in H_2$ is conjugate to a power of some $a \in \{a_1, \dots, a_n\}$.
- Let $G^{(i)}$ be the i-th commutator subgroup of G. Since H_2 is a noncyclic free group, there exists an infinite subset $I \subset \mathbb{N}$ such that $G^{(i)} \setminus G^{(i+1)}$ contains an element $x_i \in H_2$ for each $i \in I$. We may assume that each x_i is conjugate to a power of the same a. Since $G^{(i)}/G^{(i+1)}$ is torsionfree, $a \in G^{(i)} \setminus G^{(i+1)}$ for each $i \in I$. A contradiction.

Step 1 (continuation)

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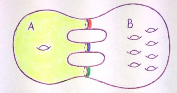

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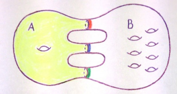
- Thus H_2 is conjugate into $H_1 = \pi_1(A)$.

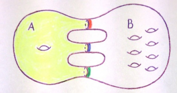
Step 2

Claim. Let $\chi(S) < 0$. Given a subgroup $H \leqslant \pi_1(S)$ that is realized by a surface $A \subset S$ such that $B := S \setminus A$ is a connected surface with $genus(B) > 0$



and given an element $g \in G$ that is not conjugate into H, there exists $H \leqslant D \overset{fin.ind.}{\leqslant} \pi_1(S)$ such that g is not conjugate into D.

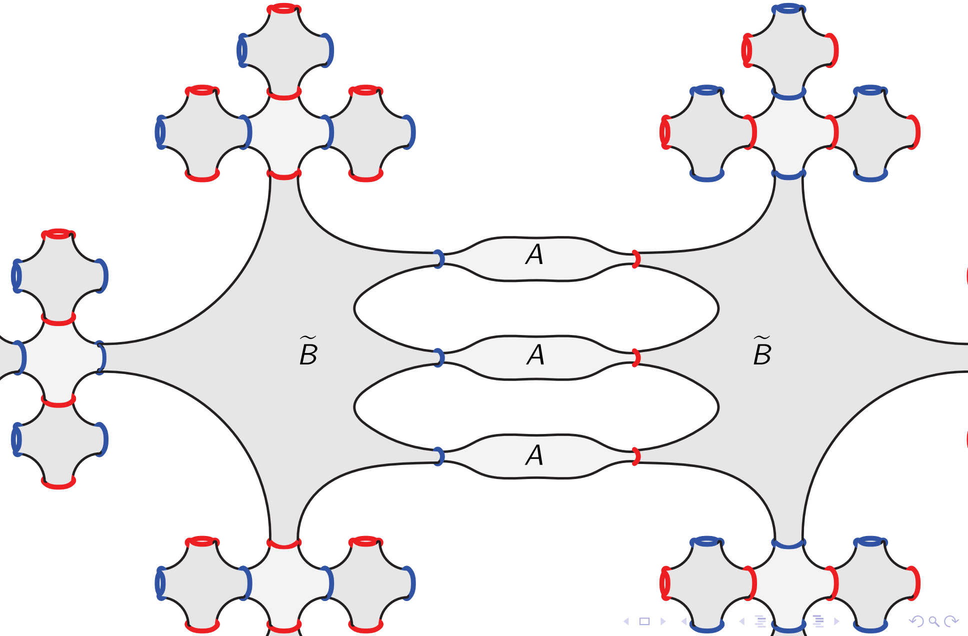
Claim. Let $\chi(S) < 0$. Given a subsurface $A \subset S$ such that $B := S \setminus A$ is a connected surface with $genus(B) > 0$



and given a loop $\gamma \subset S$ that cannot be freely homotoped into A, there exists a finitely-sheeted covering $\widetilde{S} \to S$ such that A lifts but γ does not.

Proof. We will construct such $\widetilde{S}$ by gluing several copies of special coverings of A and B.

Construction of $\widetilde{S}$ (form)



Construction of $\widetilde{S}$ (three conditions)

Endow S with a hyperbolic metric ℓ. Then all coverings of pieces of S inherit the metric ℓ. A curve is called short if its length does not exceed $\ell(\gamma)$. So, γ itself and all its lifts are short.

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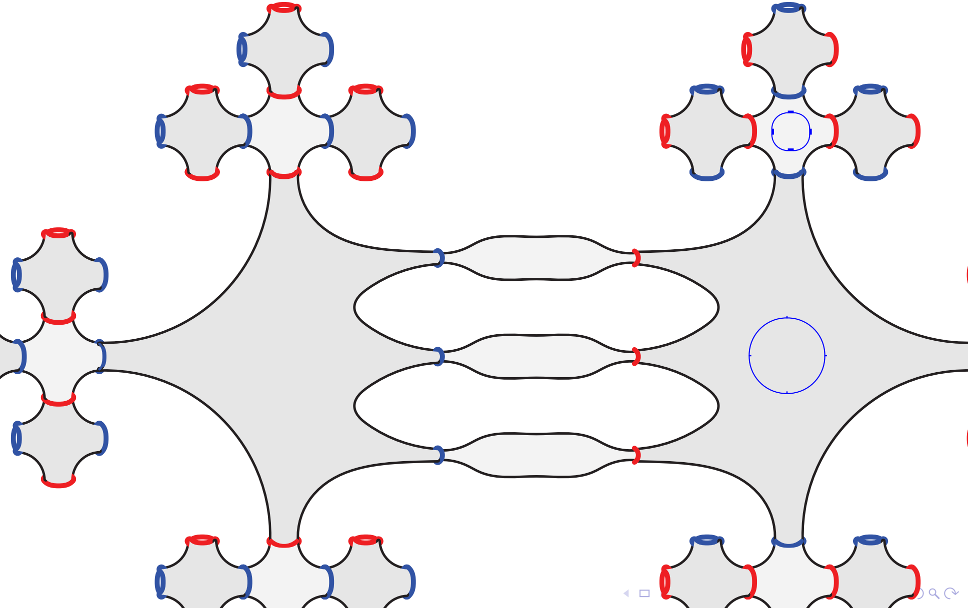
Then γ will not have closed lifts in $\widetilde{S}$ as desired.

Therefore we shall

1) put conditions on lengths of closed curves in the covering pieces,
2) put conditions on lengths of curves connecting two boundary components in each covering piece,
3) choose covering pieces so that boundaries of differen pieces under gluing have the same length).

We want: these closed curves must be long



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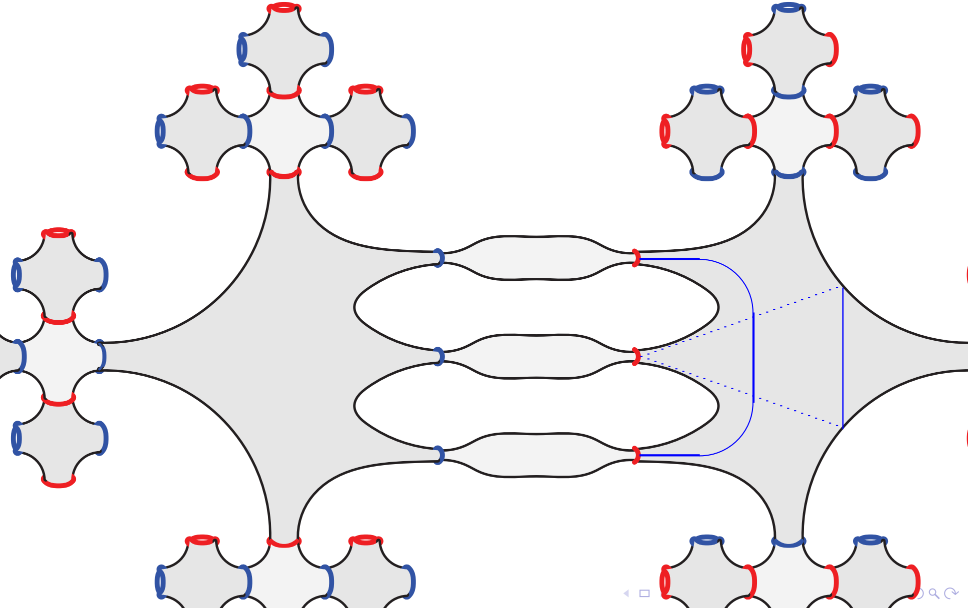
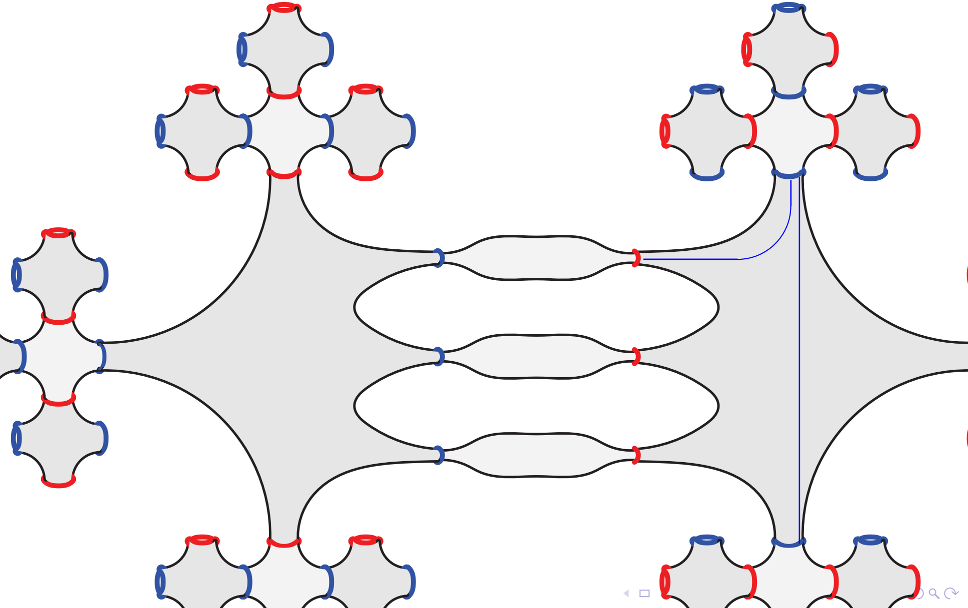


These curves are allowed to be short



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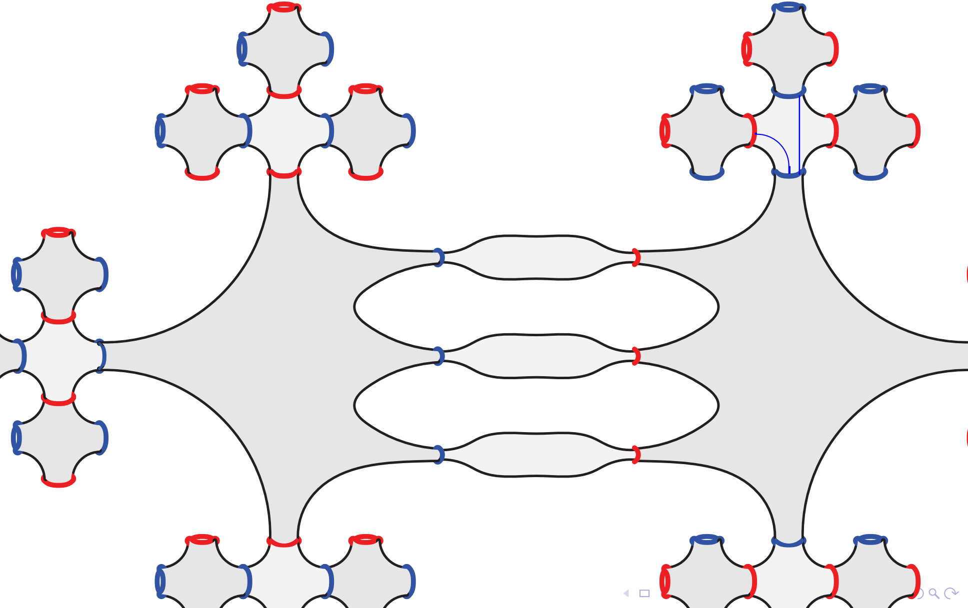
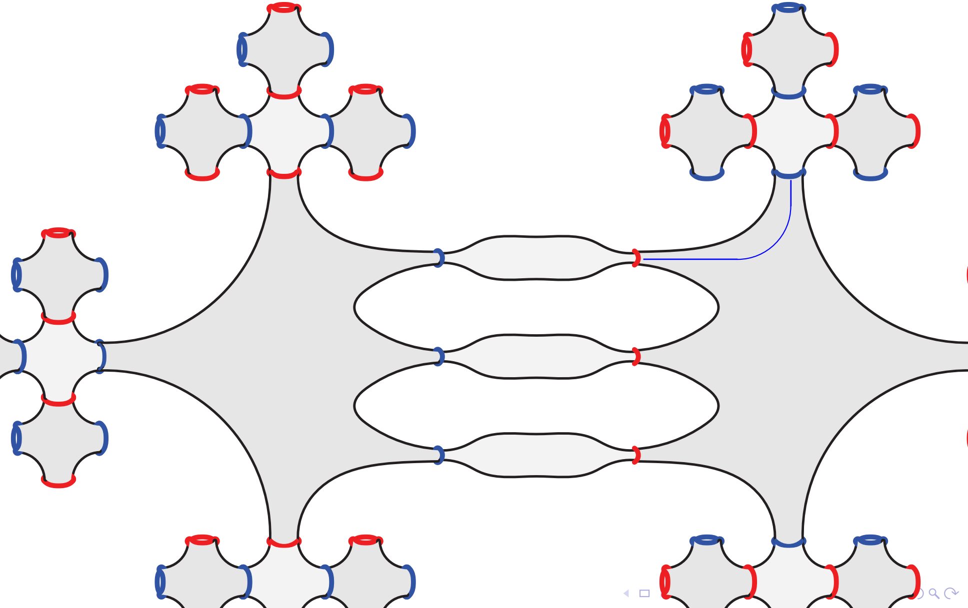


Where short curves in $\widetilde{S}$ can be?



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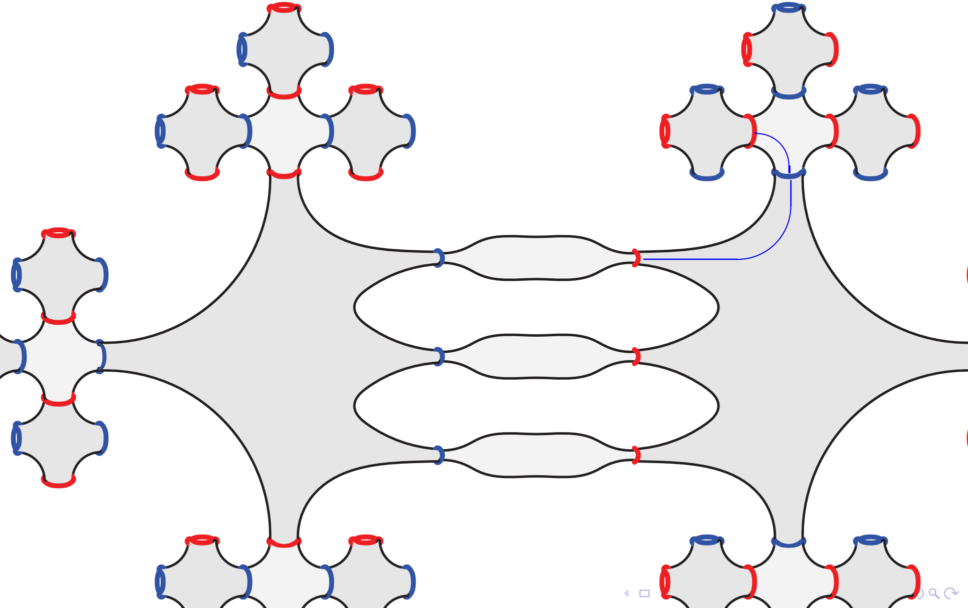

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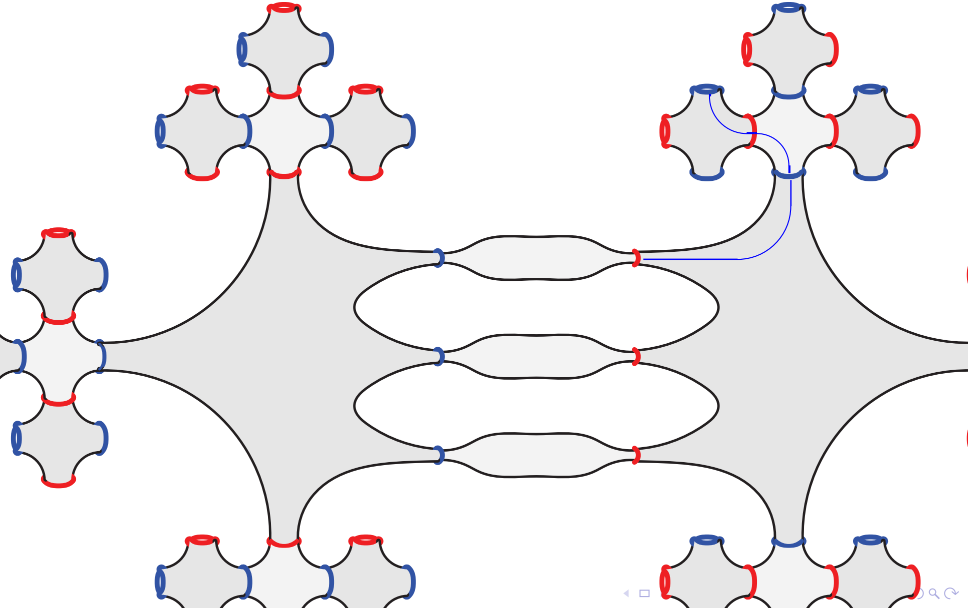
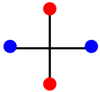
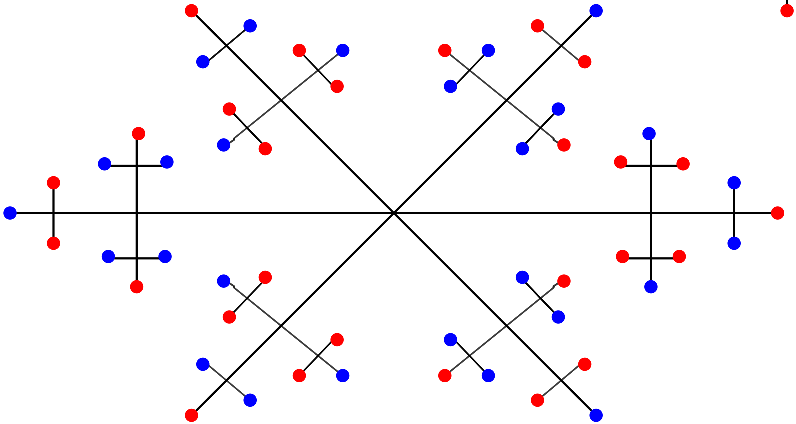


A final gluing of large and small pieces

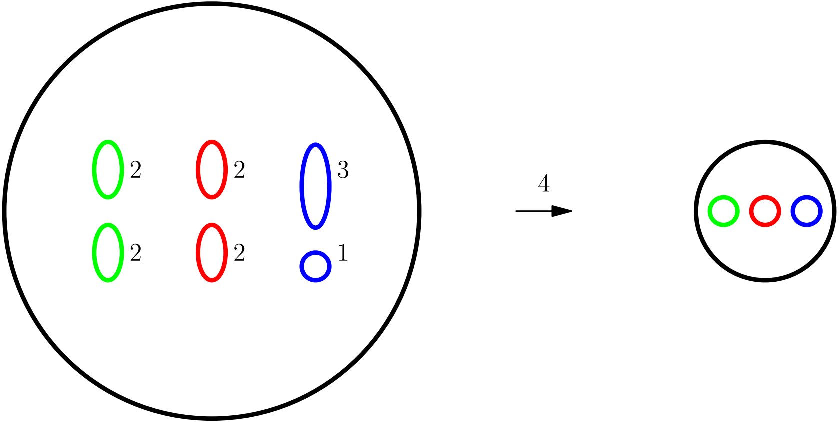




Part IV.
Hurwitz' problem

Example 1

Does such covering exist?



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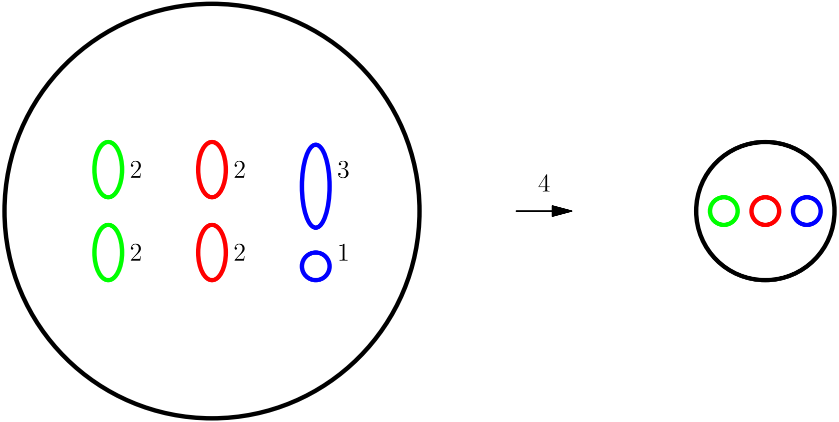
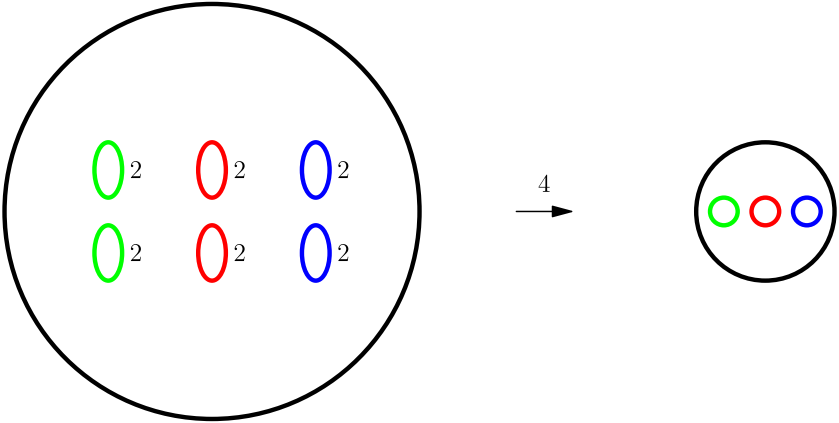


No.

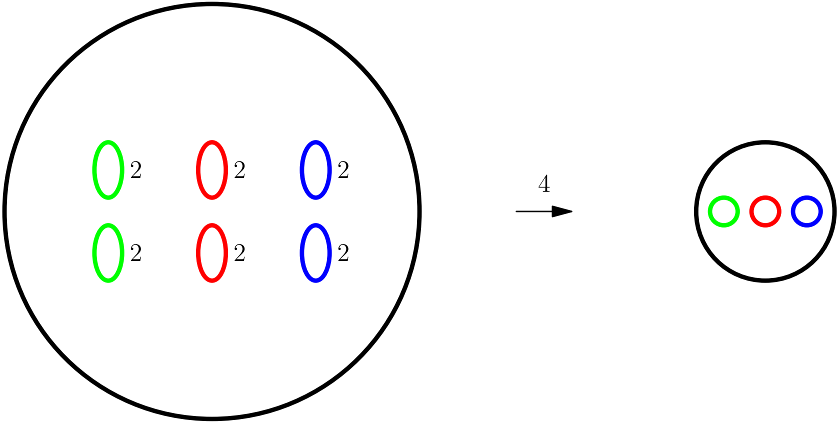
Example 2

Does such covering exist?



Example 2

Does such covering exist?



Yes.

Hurwitz' problem (formulation)

Let S be a compact surface with boundary components B_i $(i \in I)$. For which numbers d and $d_{i,1}, \ldots, d_{i,m(i)}$ $(i \in I)$, there exists a covering $\theta : \widetilde{S} \to S$ such that

1) $deg\theta = d$,

2) lifts of each boundary component B_i cover B_i with degrees $d_{i,1}, \ldots, d_{i,m(i)}$?

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There are no difficulties for $genus(S) \geqslant 1$. Partial results for $genus(S) = 0$ are in papers of Hurwitz, Husemoller, Ezell, Singerman, Edmonds, Kulkarni, Stong, Petronio, Pervova, ...

Hurwitz' problem (necessary and sufficient conditions)

Let $\pi_1(S) =$

$$\langle a_1, b_1, \ldots, a_g, b_g,\ x_1, x_2, \ldots, x_n \mid \prod_{i=1}^{g} [a_i, b_i] \cdot x_1 x_2 \ldots x_n = 1 \rangle.$$

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Theorem. There exists a covering $\widetilde{S} \to S$ of degree d with the data $\begin{pmatrix} d_{11} \\ \vdots \\ d_{1,m_1} \end{pmatrix}, \ldots, \begin{pmatrix} d_{n1} \\ \vdots \\ d_{n,m_n} \end{pmatrix}$ iff

(1) $\chi(\widetilde{S}) = d \cdot \chi(S)$,
(2) $d = d_{i1} + \cdots + d_{im_i}$ for every $i = 1, \ldots, n$,

and there exists a homomorphism $\theta : \pi_1(S) \to Perm\{1, 2, \ldots, d\}$ such that:

(3) $\mathrm{Im}(\theta)$ acts transitively on $\{1, 2, \ldots, d\}$,
(4) $\theta(x_i)$ is the product of m_i independent cycles of lengths $d_{i1}, \ldots, d_{im_i}, \quad i = 1, \ldots, n.$

Part V.
Problems on SCS and SICS

1) Are limit groups SCS?
2) Let A, B be LERF groups having a common malnormal subgroup C. Is $A *_C B$ a SCS-group (a SICS-group)?
3) Which interesting classes of groups are SCS (SICS)?
4) Investigate relations between CS, LERF, SCS, SICS.
5) Whether SCS (SICS) inherits under passing to subgroups and overgroups of finite index?
6) Which interesting classes of groups G possess the following property:
   Given fin. gen. $H_1, H_2 \leqslant G$. If each element of H_2 is conjugate into H_1, then the whole H_2 is conjugate into H_1.

THANK YOU!