Partial periodic quotients of mapping class groups

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Definition

A group $G$ is periodic if $\exists n \in \mathbb{N}$ such that $\forall g \in G$, $g^n = 1$. 
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Free Burnside group of rank $r$ and exponent $n$

$$\mathbb{B}_r(n) = \langle a_1, \ldots, a_r | x^n = 1 \rangle = \mathbb{F}_r / \mathbb{F}_r^n.$$
Theorem

Let $r \geq 2$. $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\mathbb{B}_r(n)$ is infinite.

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Theorem (Ol’shanskii-Ivanov)

Let $G$ be a hyperbolic group which is not virtually cyclic. \( \exists \kappa, n_0 \in \mathbb{N} \) such that \( \forall n \geq n_0, G/G^{\kappa n} \) is infinite.
**Question.** Given an arbitrary group $G$ and an integer $n$, what can we say about $G/G^n$?
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Example for this talk. $G = \text{Mapping class group}$.

$\Sigma$ surface of genus $g$ with $p$ punctures.

$\text{MCG}(\Sigma) = \{\text{orientation preserving homeomorphisms}\}/\{\text{isotopies}\}$
Classification of the mapping classes. (W. Thurston)

\( f \in \text{MCG}(\Sigma) \) is either

1. **periodic**: \( f \) has finite order
2. **reducible**: \( f \) permutes a collection of essential non-peripheral curves (up to isotopy) e.g. \( f \) is a Dehn Twist
3. **pseudo-Anosov**: \( f \) preserves a pair of transverse foliations and acts in an appropriate way on them.
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  a &\rightarrow b \\
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$$a \rightarrow ab$$

$$b \rightarrow a$$
Theorem 1

Let $S$ be the set of all Dehn twists in $\text{MCG}(\Sigma)$. $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ odd, $\mathbb{F}_2$ embeds in $\text{MCG}(\Sigma)/\langle \langle S^n \rangle \rangle$.

Remark. The Dehn twists generate $\text{MCG}(\Sigma)$.
Main theorems

$\Sigma$ surface of genus $g$ with $p$ punctures such that $3g + p - 3 > 1$.

**Theorem 2**

$\exists \kappa, n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ odd, there is a quotient $Q$ of $\text{MCG}(\Sigma)$ with the following properties.

1. Let $f \in \text{MCG}(\Sigma)$ pseudo-Anosov. Either $f^{\kappa n} = 1$ in $Q$ or $\exists u \in \text{MCG}(\Sigma)$ reducible or periodic such that $f^\kappa = u$ in $Q$.

2. Let $E < \text{MCG}(\Sigma)$ without pseudo-Anosov element. $\text{MCG}(\Sigma) \twoheadrightarrow Q$ induces an isomorphism from $E$ onto its image.

3. $\exists$ infinitely many $f \in Q$ which are not the image of a periodic or reducible $u \in \text{MCG}(\Sigma)$.
Dehn twists

\[ \Sigma \text{ surface of genus } g \text{ with } p \text{ punctures such that } 3g + p - 3 > 1. \]
\[ \Gamma = \pi_1(\Sigma) \text{ torsion-free hyperbolic group. } \text{MCG}(\Sigma) \simeq \text{Out}(\Gamma) \]
Dehn twists

\[ \Sigma \] surface of genus \( g \) with \( p \) punctures such that \( 3g + p - 3 > 1 \).
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**Theorem (Ol’shanskii)**

\[ \exists n_0 \in \mathbb{N}, \ \forall n \geq n_0 \text{ odd}, \ \Gamma/\Gamma^n \text{ is infinite.} \]
Dehn twists

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**Theorem (Ol’shanskii)**

$\exists n_0 \in \mathbb{N}, \forall n \geq n_0$ odd, $\Gamma/\Gamma^n$ is infinite.

**Canonical map.** $\text{MCG}(\Sigma) \to \text{Out}(\Gamma/\Gamma^n)$.

**Theorem (C.)**

$\exists n_0 \in \mathbb{N}, \forall n \geq n_0$ odd, the image of $\text{MCG}(\Gamma)$ in $\text{Out}(\Gamma/\Gamma^n)$ contains $\mathbb{F}_2$. 
Fact. $S$ set of all Dehn twists.
\[ \forall n \in \mathbb{N}, \forall f \in S, f^n = 1 \text{ in } \text{Out}(\Gamma/\Gamma^n). \]
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Example.

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\begin{align*}
  f: & \quad a \to ab & f^n: & \quad a \to ab^n \\
  b \to b & \quad b \to b
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**Commutative diagram.**

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\begin{array}{c}
\text{MCG}(\Sigma) \\
\downarrow \\
\text{MCG}(\Sigma)/\langle \langle S^n \rangle \rangle \longrightarrow \text{Out}(\Gamma/\Gamma^n)
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\]
Fact. $S$ set of all Dehn twists.
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Example.

$$f: \begin{align*}
a &\rightarrow ab \\
b &\rightarrow b
\end{align*}$$

$$f^n: \begin{align*}
a &\rightarrow ab^n \\
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Commutative diagram.

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\text{MCG}(\Sigma)/\langle\langle S^n \rangle\rangle &\rightarrow \text{Out}(\Gamma/\Gamma^n)
\end{align*}$$

Consequence. $\mathbb{F}_2$ embeds in $\text{MCG}(\Sigma)/\langle\langle S^n \rangle\rangle$. 

**General Idea.** Use the “hyperbolic features” of $\text{MCG}(\Sigma)$. 
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Complex of curves.
Simplicial complex $X$ built out of $Σ$.

- vertex : isotopy class of an essential non-peripheral curve
- $k$-simplex : collection of $k + 1$ vertices $\{\alpha_0, \ldots, \alpha_k\}$ which can be realized by disjoint curves in $Σ$. 
The complex of curves

Features of the complex of curves

\[ \Sigma \text{ surface of genus } g \text{ with } p \text{ punctures such that } 3g + p - 3 > 1. \]

\[ \text{MCG}(\Sigma) \text{ acts on } X \text{ by isometries.} \]
Features of the complex of curves

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**Theorem (Masur-Minsky)**

\( X \) is Gromov hyperbolic.
- Periodic and reducible mapping classes act *elliptically*.
- Pseudo-Anosov mapping classes act *loxodromically*.
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**Theorem (Bowditch)**

$\text{MCG}(\Sigma)$ acts *acylindrically* on $X$. 

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A more general framework

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**Remark.** Every group acts on a hyperbolic space, even properly. One needs some extra assumption.
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**Definition**

$G$ acts *acylindrically* if

$\forall d \geq 0, \exists D, N$ such that $\forall x, y \in X$ with $|x - y| \geq D$,

$$\# \{ g \in G \mid |gx - x| \leq d \quad \text{and} \quad |gy - y| \leq d \} \leq N.$$
Goal. “kill” a very large collection of $n$-th powers in $G$. 
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Construct by induction a sequence of groups $G_k$ with a "nice" action on a hyperbolic space $X_k$

\[ G = G_0 \to G_1 \to \ldots \to G_k \to G_{k+1} \to \ldots \]

\[ X = X_0 \to X_1 \to \ldots \to X_k \to X_{k+1} \to \ldots \]
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**Induction.** $G_{k+1}$ is obtained from $G_k$ by small cancellation (using the Delzant-Gromov approach).
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Induction. $G_{k+1}$ is obtained from $G_k$ by small cancellation (using the Delzant-Gromov approach).

At the limit. $Q = \lim_{\to} G_k$. 
Small cancellation theory

$G$ acts by isometries on a $\delta$-hyperbolic space $X$. $R$ set of “relations” (invariant by conjugation).
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**Small cancellation parameters.**

*Length of the pieces*

$$\Delta(R) \approx \sup_{r_1 \neq r_2} \diam(\text{Axe}(r_1) \cap \text{Axe}(r_2)) .$$

*Length of the relations*

$$T(R) = \inf_r \| r \| .$$
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Theorem (Delzant-Gromov)

\[ \exists \delta_0, \delta_1, \Delta_0, \rho \text{ with the following properties. If } \delta \leq \delta_0, \Delta(R) \leq \Delta_0 \text{ and } T(R) \geq \rho \text{ then} \]

- \[ \overline{G} = G/\langle \langle R \rangle \rangle \] acts by isometries on a \( \delta_1 \)-hyperbolic space \( \overline{X} \).
- \( \overline{G} \) inherits some properties of \( G \).
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**Remarks.**

- The constants \( \delta_0, \delta_1, \Delta_0 \) and \( \rho \) do not depend on \( G, X \) or \( R \).
- Class of groups “invariant under small cancellation”.
Controlling the small cancellation parameters

Length of the pieces.
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Length of the pieces.

Take $g, h \in G$ loxodromic. Assume that

$$\text{diam} \left( \text{Axe}(g) \cap \text{Axe}(h) \right) \gg N \max \{ \|g\|, \|h\| \} + D.$$
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Hence \( \exists i < j \) such that \( [g, h_i] = [g, h_j] \). Thus \( [g, h_{j-i}] = 1 \).

Since \( g, h \) loxodromic, \( \text{Axe}(g) = \text{Axe}(h) \).
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Partial periodic quotients of mapping class groups
Length of the relations.
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Fact: \( \exists \varepsilon > 0, \ \forall g \in G \ \text{loxodromic}, \ ||g|| \geq \varepsilon, \ \text{hence} \ ||g^n|| \approx n\varepsilon. \)
Length of the relations.

Fact: $\exists \varepsilon > 0$, $\forall g \in G$ loxodromic, $\| g \| \geq \varepsilon$, hence $\| g^n \| \gtrsim n\varepsilon$.

Consequence. if $R$ is a set of powers $g^n$ with finitely many conjugacy classes of loxodromic elements.

- $\Delta(R)$ bounded
- $T(R) \gtrsim n\varepsilon$. 
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Rescale the space $X$ by $\rho / n\epsilon$ and the small cancellation assumptions are satisfied, provided $n$ is large enough.
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Rescale the space $X$ by $\rho/n\varepsilon$ and the small cancellation assumptions are satisfied, provided $n$ is large enough.

**Difficulty.** bound the exponent $n$ at each step.
Not easy to control the parameters of acylindricity: there are more and more small isometries.
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**Burnside groups of odd exponents.** At each step, every elementary subgroup is cyclic.
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**Burnside groups of odd exponents.** At each step, every elementary subgroup is cyclic.

**Partial quotient of mapping class groups.** Elementary subgroups can be very large: they contain $\mathbb{Z}^m$. 
Invariants for the group actions.

1. \( r_{inj}(G, X) = \inf \{ \|g\| \mid g \in G \text{ loxodromic} \} \)
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1. $r_{\text{inj}}(G, X) = \inf \{ \| g \| \mid g \in G \text{ loxodromic} \}$

2. $\nu = \nu(G, X)$ smallest integer $p$ with the following property. Given $g, h \in G$ with $h$ loxodromic, if $\langle g, hgh^{-1}, \ldots, h^pgh^{-p} \rangle$ is elliptic then $\langle g, h \rangle$ is elementary.
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3. $A(G, X) = \sup \text{diam} (\text{Axe}(g_0) \cap \cdots \cap \text{Axe}(g_\nu))$, where $\|g_i\| \leq 1000\delta$ and $\langle g_0, \ldots, g_\nu \rangle$ non-elementary.
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Remark. The parameters speak about the small scale properties of the group action.
Assume that $G$ has no involution. Let $g, h \in G$, loxodromic. If $\text{Axe}(g) \neq \text{Axe}(h)$ is not elementary then

$$\text{diam}(\text{Axe}(g) \cap \text{Axe}(h)) \leq (\nu + 1) \max\{\|g\|, \|h\|\} + A(G, X) + \delta$$

It allows to bound the length of the pieces from above.
Key lemma

Assume that $G$ has no involution. Let $g, h \in G$, loxodromic. If $\text{Axe}(g) \neq \text{Axe}(h)$ is not elementary then

$$\text{diam}(\text{Axe}(g) \cap \text{Axe}(h)) \leq (\nu + 1) \max \{ \|g\|, \|h\| \} + A(G, X) + \delta$$

It allows to bound the length of the pieces from above.

Other important point. The invariants of the action refer to the geometry at a small scale. One can control the invariants of $(\overline{G}, \overline{X})$ using the one of $(G, X)$. 
General result and applications

Start with $G$ torsion-free acting acylindrically on a hyperbolic length space $X$. 
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Iterate small cancellation.

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G = G_0 \to G_1 \to \ldots \to G_k \to G_{k+1} \to \ldots
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X = X_0 \to X_1 \to \ldots \to X_k \to X_{k+1} \to \ldots
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At each step
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At each step
- $G_k$ acts acylindrically on $X_k$ with controlled invariants,
Start with $G$ torsion-free acting acylindrically on a hyperbolic length space $X$.

Iterate small cancellation.

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At each step

- $G_k$ acts acylindrically on $X_k$ with controlled invariants,
- kill $n$-th power of some loxodromic elements,
Start with $G$ torsion-free acting acylindrically on a hyperbolic length space $X$.

Iterate small cancellation.

$G = G_0 \twoheadrightarrow G_1 \twoheadrightarrow \ldots \twoheadrightarrow G_k \twoheadrightarrow G_{k+1} \twoheadrightarrow \ldots$

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At each step

- $G_k$ acts acylindrically on $X_k$ with controlled invariants,
- kill $n$-th power of some loxodromic elements,
- $G_k \twoheadrightarrow G_{k+1}$ restricted to elliptic subgroups is one-to-one.
Theorem

Let $G$ be a group acting acylindrically on a hyperbolic length space $X$. Assume that $G$ is torsion-free and non-elementary. There exists $n_0$ such that for all $n \geq n_0$ odd, there is a quotient $Q$ of $G$ with the following properties.

- Let $g \in Q$. Either $g^n = 1$ or there exists $u \in G$ elliptic such that $g = u$ in $Q$.
- Let $E \leq G$ elliptic. $G \to Q$ induces an isomorphism from $E$ onto its image.
- There exist infinitely many elements in $Q$ which are not the image of an elliptic element of $G$. 
Remarks.
One can weaken the assumption to allow

- $G$ to have odd torsion
- $G$ to have parabolic isometries for the action on $X$ (the action is no more acylindrical).
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Application to the mapping class group.
$\text{MCG}(\Sigma)$ is not torsion-free!! It has even torsion.
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One can weaken the assumption to allow

- $G$ to have odd torsion
- $G$ to have parabolic isometries for the action on $X$ (the action is no more acylindrical).

Application to the mapping class group.

$\text{MCG}(\Sigma)$ is not torsion-free!! It has even torsion. There exists a finite-index subgroup $H$ of $\text{MCG}(\Sigma)$ which is torsion-free. Apply the theorem with $H$. 
Other applications.

**Theorem**

Let $A$ and $B$ be two groups without involution. Let $C$ be a subgroup of $A$ and $B$ malnormal in $A$ or $B$. There exists a quotient $Q$ of $A \ast_C B$ with the following properties.

1. The groups $A$ and $B$ embed into $Q$.
2. For all $g \in Q$, if $g$ is not conjugated to an element of $A$ or $B$, then $g^n = 1$.
3. There exist infinitely many elements in $Q$ which are not conjugated to an element of $A$ or $B$. 