Partial periodic quotients of mapping class groups

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Burnside problem

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Free Burnside group of rank r and exponent n

$$\mathbb{B}_r(n) = \langle a_1, \dots, a_r | x^n = 1 \rangle = \mathbb{F}_r / \mathbb{F}_r^n.$$

Theorem

Let $r \geqslant 2$. $\exists n_0 \in \mathbb{N}$ such that $\forall n \geqslant n_0$, $\mathbb{B}_r(n)$ is infinite.

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Theorem (Ol'shanskiĭ-Ivanov)

Let G be a hyperbolic group which is not virtually cyclic.
$\exists \kappa, n_0 \in \mathbb{N}$ such that $\forall n \geqslant n_0$, $G/G^{\kappa n}$ is infinite.

Question. Given an arbitrary group G and an integer n, what can we say about G/G^n?

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Example for this talk. $G =$ Mapping class group.

Σ surface of genus g with p punctures.

$$\mathrm{MCG}(\Sigma) = \{\text{orientation preserving homeomorphisms}\}/\{\text{isotopies}\}$$

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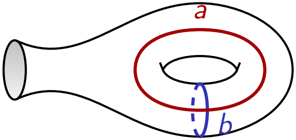

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1. *periodic*: f has finite order
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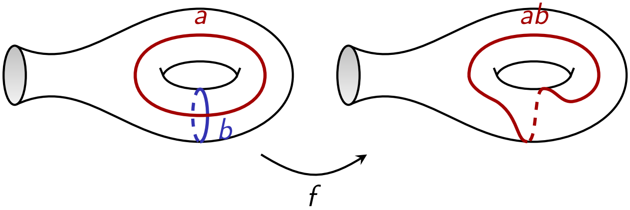

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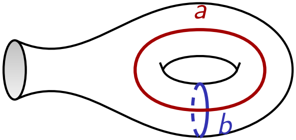
1. *periodic*: f has finite order
2. *reducible*: f permutes a collection of essential non-peripheral curves (up to isotopy) e.g. f is a Dehn Twist
3. *pseudo-Anosov*: f preserves a pair of transverse foliations and acts in an appropriate way on them.

$$\begin{aligned} a &\to ab \\ b &\to a \end{aligned}$$



Main theorems

Σ surface of genus g with p punctures such that $3g + p - 3 > 1$.

Theorem 1

Let S be the set of all Dehn twists in $\mathrm{MCG}(\Sigma)$. $\exists n_0 \in \mathbb{N}$ such that $\forall n \geqslant n_0$ odd, $\mathbb{F}_2$ embeds in $\mathrm{MCG}(\Sigma)/\langle\langle S^n \rangle\rangle$.

Remark. The Dehn twists generate $\mathrm{MCG}(\Sigma)$.

Main theorems

Σ surface of genus g with p punctures such that $3g + p - 3 > 1$.

Theorem 2

$\exists \kappa, n_0 \in \mathbb{N}$ such that $\forall n \geqslant n_0$ odd, there is a quotient Q of $\mathrm{MCG}(\Sigma)$ with the following properties.

1. Let $f \in \mathrm{MCG}(\Sigma)$ pseudo-Anosov. Either $f^{\kappa n} = 1$ in Q or $\exists u \in \mathrm{MCG}(\Sigma)$ reducible or periodic such that $f^{\kappa} = u$ in Q.
2. Let $E < \mathrm{MCG}(\Sigma)$ without pseudo-Anosov element. $\mathrm{MCG}(\Sigma) \twoheadrightarrow Q$ induces an isomorphism from E onto its image.
3. $\exists$ infinitely many $f \in Q$ which are not the image of a periodic or reducible $u \in \mathrm{MCG}(\Sigma)$.

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Canonical map. $\mathrm{MCG}(\Sigma) \to \mathrm{Out}(\Gamma/\Gamma^n)$.

Theorem (C.)

$\exists n_0 \in \mathbb{N}$, $\forall n \geqslant n_0$ odd, the image of $\mathrm{MCG}(\Gamma)$ in $\mathrm{Out}(\Gamma/\Gamma^n)$ contains $\mathbb{F}_2$.

Fact. S set of all Dehn twists.
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$$f: \quad a \to ab \qquad\qquad f^n: \quad a \to ab^n$$
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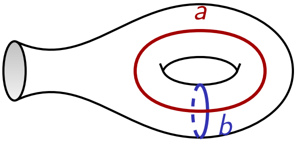

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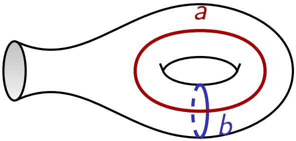

Commutative diagram.

$$\begin{array}{ccc} \mathrm{MCG}(\Sigma) & & \\ \downarrow & \searrow & \\ \mathrm{MCG}(\Sigma)/\langle\langle S^n \rangle\rangle & \longrightarrow & \mathrm{Out}\,(\Gamma/\Gamma^n) \end{array}$$

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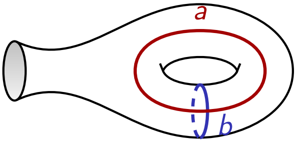

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Consequence. $\mathbb{F}_2$ embeds in $\mathrm{MCG}(\Sigma)/\langle\langle S^n \rangle\rangle$.

The complex of curves

General Idea. Use the “hyperbolic features” of $MCG(\Sigma)$.

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Complex of curves.
Simplicial complex X built out of Σ.

- vertex : isotopy class of an essential non-peripheral curve
- k-simplex : collection of $k+1$ vertices $\{\alpha_0, \dots, \alpha_k\}$ which can be realized by disjoint curves in Σ.

Features of the complex of curves

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Theorem (Bowditch)

$\mathrm{MCG}(\Sigma)$ acts *acylindrically* on X.

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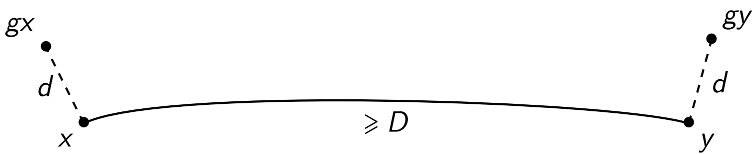
Definition

G acts *acylindrically* if
$\forall d \geqslant 0$, $\exists D, N$ such that $\forall x, y \in X$ with $|x - y| \geqslant D$,

$$\#\{g \in G \mid |gx - x| \leqslant d \quad \text{and} \quad |gy - y| \leqslant d\} \leqslant N.$$



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Construct by induction a sequence of groups G_k with a "nice" action on a hyperbolic space X_k

$$\begin{array}{ccccccccccc} G = G_0 & \twoheadrightarrow & G_1 & \twoheadrightarrow & \dots & G_k & \twoheadrightarrow & G_{k+1} & \twoheadrightarrow & \dots \\ \circlearrowleft & & \circlearrowleft & & & \circlearrowleft & & \circlearrowleft & & \\ X = X_0 & \to & X_1 & \to & \dots & X_k & \to & X_{k+1} & \to & \dots \end{array}$$

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At the limit. $Q = \varinjlim G_k$.

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G acts by isometries on a δ-hyperbolic space X. R set of "relations" (invariant by conjugation).

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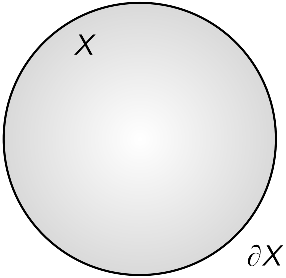
Small cancellation parameters.

Length of the pieces

$$\Delta(R) \approx \sup_{r_1 \neq r_2} \operatorname{diam}\left(\operatorname{Axe}(r_1) \cap \operatorname{Axe}(r_2)\right).$$

Length of the relations

$$T(R) = \inf_r \|r\|.$$



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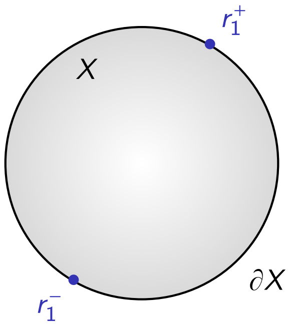

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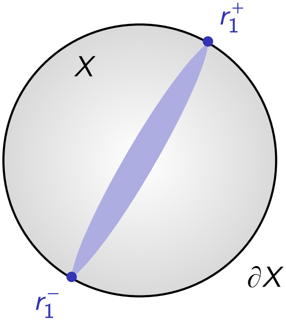

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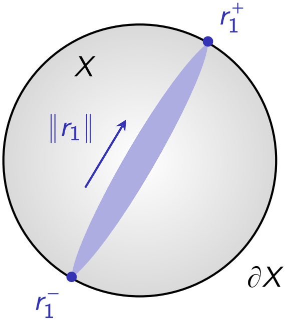

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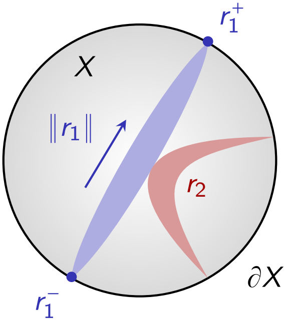

Theorem (Delzant-Gromov)

$\exists \delta_0, \delta_1, \Delta_0, \rho$ with the following properties. If $\delta \leqslant \delta_0$, $\Delta(R) \leqslant \Delta_0$ and $T(R) \geqslant \rho$ then

- $\overline{G} = G/\langle\langle R \rangle\rangle$ acts by isometries on a δ_1-hyperbolic space $\overline{X}$.
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Remarks.

- The constants δ_0, δ_1, Δ_0 and ρ do not depend on G, X or R.
- Class of groups "invariant under small cancellation".

Controlling the small cancellation parameters

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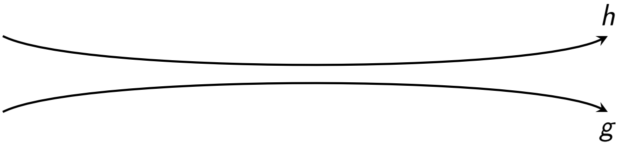
Take $g, h \in G$ loxodromic. Assume that

$$\operatorname{diam}\left(\operatorname{Axe}(g) \cap \operatorname{Axe}(h)\right) \gg N \max\{\|g\|, \|h\|\} + D.$$



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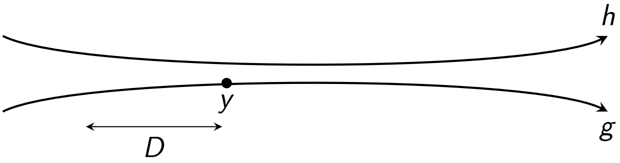

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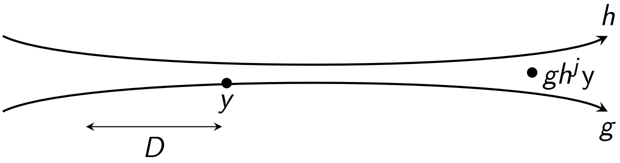

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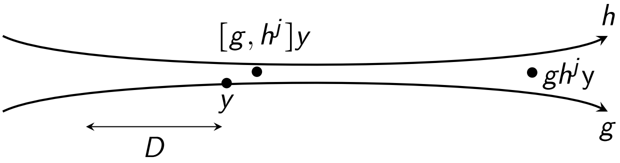

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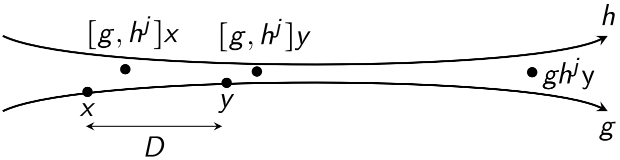

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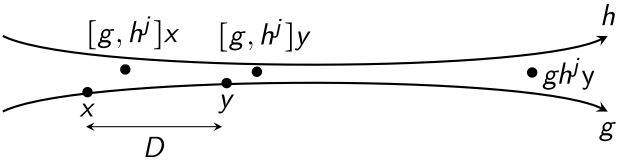

Hence $\exists i < j$ such that $[g,h^i] = [g,h^j]$. Thus $[g,h^{j-i}] = 1$. Since g, h loxodromic, $\operatorname{Axe}(g) = \operatorname{Axe}(h)$.

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Difficulty. bound the exponent n at each step.

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Burnside groups of odd exponents. At each step, every elementary subgroup is cyclic.

Partial quotient of mapping class groups. Elementary subgroups can be very large: they contain $\mathbb{Z}^m$.

Invariants for the group actions.

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3. $A(G, X) = \sup \operatorname{diam}(\operatorname{Axe}(g_0) \cap \dots \cap \operatorname{Axe}(g_\nu))$, where $\|g_i\| \leqslant 1000\delta$ and $\langle g_0, \dots, g_\nu\rangle$ non-elementary.

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3. $A(G,X) = \sup \operatorname{diam}\left(\operatorname{Axe}(g_0) \cap \dots \cap \operatorname{Axe}(g_\nu)\right)$, where $\|g_i\| \leqslant 1000\delta$ and $\langle g_0, \dots, g_\nu \rangle$ non-elementary.

Remark. The parameters speak about the small scale properties of the group action.

Key lemma

Assume that G has *no involution*. Let $g, h \in G$, loxodromic. If $\text{Axe}(g) \neq \text{Axe}(h)$ is not elementary then

$$\text{diam}(\text{Axe}(g) \cap \text{Axe}(h)) \leqslant (\nu + 1) \max\{\|g\|, \|h\|\} + A(G, X) + \delta$$

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Other important point. The invariants of the action refer to the geometry at a small scale. On can control the invariants of $(\overline{G}, \overline{X})$ using the one of (G, X).

General result and applications

Start with G torsion-free acting acylindrically on a hyperbolic length space X.

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Iterate small cancellation.

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At each step

- G_k acts acylindrically on X_k with controlled invariants,
- kill n-th power of some loxodromic elements,
- $G_k \twoheadrightarrow G_{k+1}$ restricted to elliptic subgroups is one-to-one.

Theorem

Let G be a group acting acylindrically on a hyperbolic length space X. Assume that G is torsion-free and non-elementary. $\exists n_0$ such $\forall n \geqslant n_0$ odd, there is a quotient Q of G with the following properties.

- Let $g \in Q$. Either $g^n = 1$ or $\exists u \in G$ elliptic such that $g = u$ in Q.
- Let $E < G$ elliptic. $G \twoheadrightarrow Q$ induces an isomorphism from E onto its image.
- $\exists$ infinitely many elements in Q which are not the image of an elliptic element of G.

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One can weaken the assumption to allow

- G to have odd torsion
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Application to the mapping class group.
$\mathrm{MCG}(\Sigma)$ is not torsion-free!! It has even torsion.
There exists a finite-index subgroup H of $\mathrm{MCG}(\Sigma)$ which is torsion-free. Apply the theorem with H.

Other applications.

Theorem

Let A and B be two groups without involution. Let C be a subgroup of A and B malnormal in A or B.
$\exists n_0$ such that $\forall n \geqslant n_0$ odd, there exists a quotient Q of $A *_C B$ with the following properties.

1. The groups A and B embed into Q.
2. $\forall g \in Q$, if g is not conjugated to an element of A or B then $g^n = 1$.
3. $\exists$ infinitely many elements in Q which are not conjugated to an element of A or B.