Cayley graphs of relatively hyperbolic groups

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This talk is based in a joint work with Laura Ciobanu:

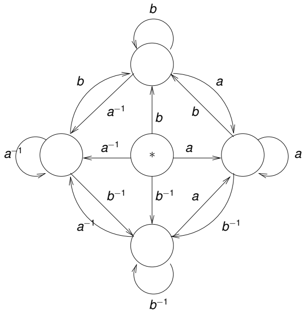
Finite generating sets of relatively hyperbolic groups and applications to geodesic languages. `arXiv:1402.2985`

Outline

1. Regular Geodesics and FFTP
2. Regular Conjugacy geodesics and BCD
3. Generating sets of relatively hyperbolic groups
4. Relatively hyperbolic groups with FFTP
5. Relatively hyperbolic groups with BCD or NSC

The language of geodesics of a free group is regular

Let $\mathbf{F} = \langle a, b \mid \, \rangle$ and $X = \{a, b\}^{\pm 1}$. An automaton recognizing $\mathrm{Geo}(\mathbf{F}, X)$.



$$\mathrm{Geo}(\mathbf{F}, X) = [(X^* aAX^*) \cup (X^* AaX^*) \cup (X^* bBX^*) \cup (X^* BbX^*)]^c$$

The falsification by fellow traveler property

FFTP

A Cayley graph $\Gamma(G, X)$ has k-FFTP if every non-geodesic path k-fellow travels with a shorter path with same end points.

Building an automaton for the language of geodesics

If $\Gamma(G, X)$ has k-FFTP, we do not need to remember the whole path to know if we are following a geodesics. Suppose we know

- all possible positions of a companion with respect to our actual position (a ball of radius k in the Cayley graph)
- together with the time difference for reaching our position and the time the companion needed to reach its position. Notice that if we are on a geodesic, the time differences can not exceed k.

Use all that information to build an automaton (at most $2k^{|\mathbb{B}(k)|}$ states).

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Cayley graphs with FFTP

FFTP depends on the generating set

There is a virtually abelian group such for which some Cayley graphs have FFTP and some don't.

Examples

- Abelian and Hyperbolic groups have FFTP with respect to any generating set.
- Virtually abelian and geometrically finite hyperbolic groups have FFTP with respect to some generating sets [Neumann-Shapiro].
- Coxeter groups [Noskov], Garside groups [Holt], Artin groups of large type [Holt-Rees] with respect to the standard generators.
- Groups with certain group actions of Buildings and CAT(0) Cubical Complexes [Noskov].

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Properties of (G, X) with FFTP.

Properties

Let G be a group, X a finite generating and such that (G, X) has FFTP.

- (G, X) has a regular language of geodesic words. (G, X) has finitely many cone types [Neumann, Shapiro]
- G has a finite presentation with a Dehn function that is at most quadratic [Elder]
- G is of type F_3 [Elder].

For simplicity, we say that G has FFTP if there is some finite generating set X such that (G, X) has FFTP.

Theorem (Antolin, Ciobanu)

If G is hyperbolic relative to groups with FFTP, then G has FFTP.

Corollary

FFTP is preserved under free products with finite amalgamation and HNN extensions with finite associate subgroups.

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Groups hyperbolic relative to virtually abelian have FFTP.

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Conjugacy in free groups

Let $\mathbb{F} = \langle a, b \mid \rangle$ be a free group of rank 2.

$$AbaBaBa$$

$$aBBaabA$$

A *conjugacy geodesic* over (G, X) is a geodesic word that has minimal length among the elements in its conjugacy class. The language of conjucagy geodesics in a $(\mathbb{F}, X = \{a, b, A, B\})$ is regular.

$$\mathrm{ConjGeo}(\mathbf{F}, X) = \mathrm{Geo}(\mathbf{F}, X) \cap (aX^*A)^c \cap (AX^*a)^c \cap (bX^*B)^c \cap (bX^*B)^c$$

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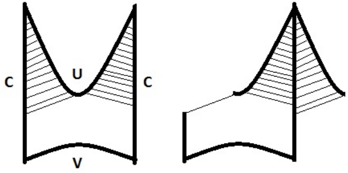
Conjugacy in Hyperbolic Cayley graphs

Suppose that $\Gamma(G, X)$ is δ-hyperbolic.
Let U and V geodesics over X, $\ell(U) >> \delta$, $CUC^{-1} =_G V$.

- U is in the 2δ-neighbourhood of the other three sides.
- If $\mathcal{N}_{2\delta}(U) \cap V = \emptyset$, U is not a cyclic geodesic.



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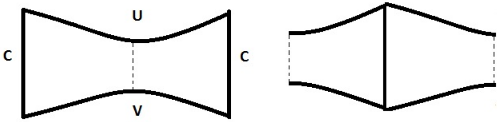

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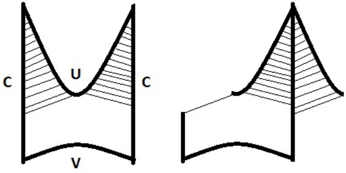

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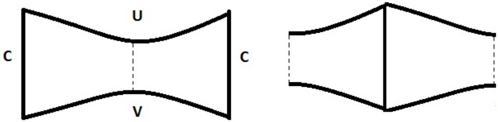

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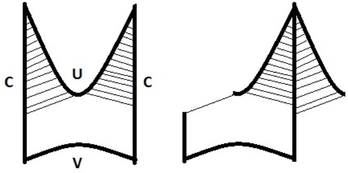

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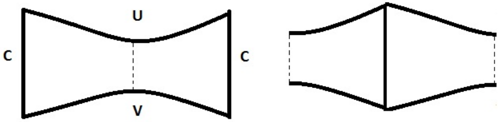

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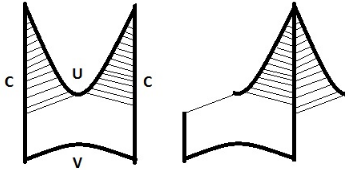

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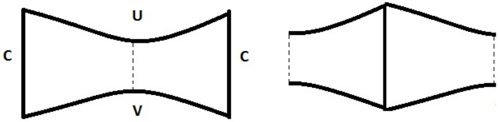

BCD

Let $A \geq 0$. We say that (G, X) has A-BCD (Bounded conjugacy diagrams) if for every $U, V \in \mathrm{CycGeo}(G, X)$, $gUg^{-1} =_G V$ for some $g \in G$, there exists $h \in G$ and cyclic shifts U' and V' of U and V such that $hU'h^{-1}_G = V'$ and

$$\min\{\max\{\ell(U), \ell(V)\}, |h|_X\} \leq A.$$

Examples

- Free groups have 0-BCD with respect to free basis.
- Hyperbolic groups have $(8\delta + 1)$-BCD.
- Abelian groups have 0-BCD.

Theorem (Antolin, Ciobanou)

Let G be hyperbolic relative to abelian groups. There is finite generating set X of G and $A \geq 0$ such that (G, X) has A-FFTP and A-BCD.

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NSC

Neighboring shorter conjugate property

(G, X) has A-NSC if for any $U \in \text{CycGeo}(G, X) - \text{ConjGeo}(G, X)$, there exists words C and V and a cyclic permutation U' of U, such that $\ell(V) < \ell(U)$, $\ell(C) \leq A$ and $CU'C^{-1} =_G V$.

A-BCD implies A-NSC

Lemma (Ciobanu, Hermiller, Holt, Rees)

If (G, X) has A-FFTP and A-NSC, $\text{ConjGeo}(G, X)$ *is regular.*

Skecth of the proof

With a the same idea as for FFTP we build automatons accepting

$$\mathcal{L}_g = \{W \in \text{Geo}(G, X) \mid |gWg^{-1}|_X \geq |W|_X\}$$

$$\text{ConjGeo}(G, X) = (\text{Cyc}(\text{Geo}(G, X)^c))^c \cap (\cup_{|g|_X \leq A} \mathcal{L}_g)^c.$$

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Lemma (Ciobanu, Hermiller, Holt, Rees)

If (G, X) has A-FFTP and A-NSC, $\text{ConjGeo}(G, X)$ is regular.

Theorem (Antolin, Ciobanu)

Let G be hyperbolic relative to virtually abelian groups. There is finite generating set X of G and $A \geq 0$ such that (G, X) has A-FFTP and A-NSC. In particular, $\text{ConjGeo}(G, X)$ is regular.

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Relatively hyperbolic groups

Let G be a group, $\{H_\omega\}_{\omega\in\Omega}$ a family of subgroups and $\mathcal{H} = \cup H_\omega$.

G is hyperbolic relative to $\{H_\omega\}_{\omega\in\Omega}$, if there is a finite subset $X \subseteq G$ such that

- $\pi\colon F = \langle X \mid \rangle * (*_{\omega\in\Omega} H_\omega) \to G$ is surjective
- $\ker(\pi) = \langle R^F \rangle$ for a finite set R of F.
- there is D such that for every $f \in F$ such that $\pi(f) = 1$,

$$f = \prod_{i=1}^{D|f|_{X\cup\mathcal{H}}} f_i r_i f_i^{-1},$$

$r_i \in R$, $f_i \in F$.

The Cayley graph $\Gamma(G, X \cup \mathcal{H})$ is δ-hyperbolic.

Relating generating sets

A word W over X has *trivial shortenings* if contains subwords in some $X \cap H_\omega$ that are not geodesics.

Construction

To a word W over X with no trivial shortenings we associated a word $\widehat{W}$ over $X \cup \mathcal{H}$, $W =_G \widehat{W}$ in the following way.

- $S(W) = \{U \text{ subword of } W \mid U \in (X \cap H_\omega)^* \text{ for some } \omega \in \Omega\}$
- Choose $V \in S(W)$, $\ell(V) = \max_{U \in S(W)} \ell(U)$.
- If $\ell(V) \leq 1$, we put $\widehat{W} \equiv W$.
- If $\ell(V) > 1$, $V =_G h \in H_\omega$, and $W \equiv AVB$, $\widehat{W} \equiv \widehat{A}h\widehat{B}$.

Lemma

There is a finite subset $\mathcal{H}_0 \subseteq \mathcal{H}$, such that for any subset $\mathcal{H}_0 \subseteq \mathcal{H}' \subseteq \mathcal{H}$ and for any 2-local geodesic word with no trivial shortenings W over $X \cup \mathcal{H}'$, $\widehat{W}$ is a 2-local geodesic.

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Lemma

There is a finite set of non-geodesic words $\widehat{\Phi}$ over $X \cup \mathcal{H}$, such that if $W \in (X \cup \mathcal{H})^*$ is 2-local geodesic and does not contain subwords of $\widehat{\Phi}$ then W labels a (λ, c)-quasi-geodesic in $\Gamma(G, X \cup \mathcal{H})$.

Idea of the proof.

There is a $k > 0$ such that every k-local geodesic is a (λ, c)-quasi-geodesic.
Let Δ be the set of all the relations in $\Gamma(G, X \cup \mathcal{H})$ of length at most $2k$ and with all components isolated.
The set Δ is finite, and a 2-local geodesic path is k-local geodesic if its label does not contain more than half of some relation in Δ. □

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Lemma (Generating Set Lemma)

There is a finite subset $\mathcal{H}'$ of $\mathcal{H}$ such that for every finite subset Y of G

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there is a finite subset Φ of non-geodesic words over Y such that if W has no trivial shortenings and does not contain words of Φ as subwords then $\widehat{W}$ is a 2-local geodesic (λ, c)-quasi-geodesic.

Proof.

Let $\mathcal{H}'$ satisfying that $\mathcal{H}_0 \subseteq \mathcal{H}$ and for every word $V \in \widehat{\Phi}$ there is a shorter $U \in (X \cup \mathcal{H}')^*$, $U =_G V$. Fix Y and take Φ the set of words V with non trivial shortenings such that $\widehat{V} \in \widehat{\Phi}$.

- Φ is a finite set of non-geodesic words.
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1. Regular Geodesics and FFTP
2. Regular Conjugacy geodesics and BCD
3. Generating sets of relatively hyperbolic groups
4. Relatively hyperbolic groups with FFTP
5. Relatively hyperbolic groups with BCD or NSC

Theorem (Antolin, Ciobanu)

If G is hyperbolic relative to groups $\{H_\omega\}_{\omega\in\Omega}$ with FFTP, then G has FFTP.

Lemma

If H has FFTP, then any finite generating set Y can be completed to a finite set Z, such that (H, Z) is FFTP.

From the Generating Set Lemma and the previous Lemma, G admits a finite generating set X satisfying:

- $(H_\omega, H_\omega \cap X)$ is FFTP
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Let W be any word over X.

- There is K_1 (depending on $(H_\omega, X \cap H_\omega)$ such that if W has trivial shortenings, then it K_1-fellow travels with a shorter word.
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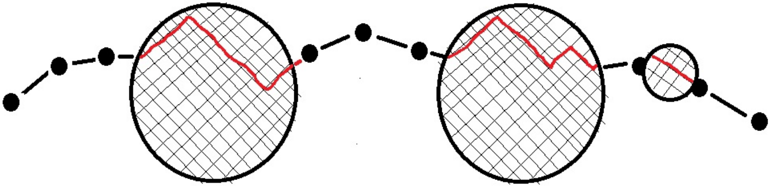

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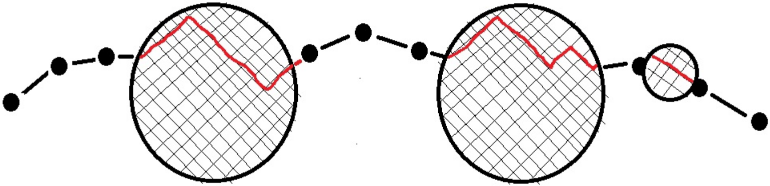

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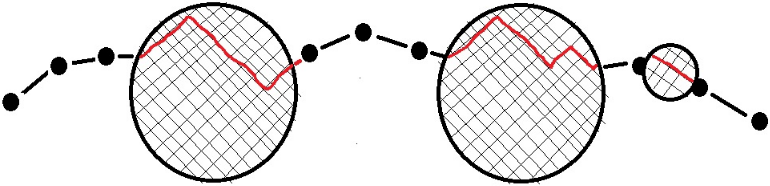

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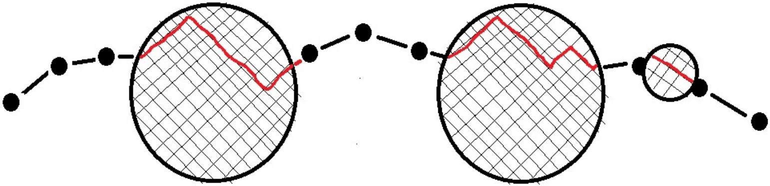

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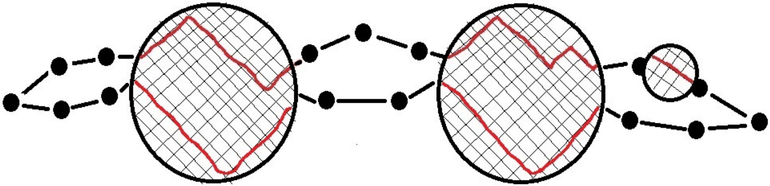

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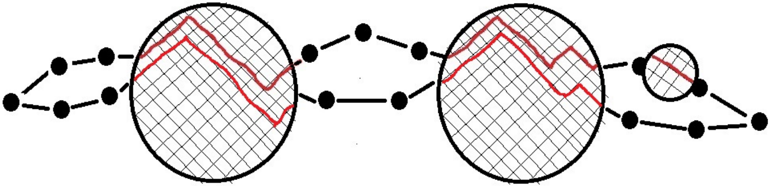

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There exists a finite subset $\mathcal{H}' \subseteq \mathcal{H}$ such that, for every finite symmetric generating set X of G satisfying that

(i) $Y \cup \mathcal{H}' \subseteq X \subseteq Y \cup \mathcal{H}$ *and*

(ii) $(H_\omega, H_\omega \cap X)$ *has the bounded conjugacy diagrams property (resp. neighboring shorter conjugate property) for all* $\omega \in \Omega$,

then the pair (G, X) has the bounded conjugacy diagrams property (resp. neighboring shorter conjugate property).
Moreover if

(iii) $(H_\omega, H_\omega \cap X)$ *has the FFTP for all* $\omega \in \Omega$,

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If H is virtually abelian, any finite generating set can be completed to have FFTP and NSC.

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Lemma

There is $K > 0$, such that for any pair of cyclic (λ, c)-quasigeodesics U, V over $X \cup \mathcal{H}$ that are conjugate, there exists cyclic permutations U' of U, V' of V and C a geodesic word over $X \cup \mathcal{H}$ such that

$$CU'C^{-1} =_G V'$$

and

$$\min\{\max\{\ell(U), \ell(V)\}, \ell(C)\} < K.$$

- From the Generating Set Lemma, we can assume that if W is a cyclic geodesic, then $\widehat{W}$ is a cyclic (λ, c)-quasigeodesic.
- The proof of the Theorem follows from combining the previous Lemma, analyizing minimal conjugacy diagrams and the following Lemma of Osin.

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Lemma (Osin)

There exists $D = D(G, X, \lambda, c) > 0$ such that the following hold. Let $\mathcal{P} = p_1 p_2 \dots p_n$ be an n-gon and I a distinguished subset of sides of $\mathcal{P}$ such that if $p_i \in I$, p_i is an isolated component in $\mathcal{P}$, and if $p_i \notin I$, p_i is a (λ, c)-quasi-geodesic. Then

$$\sum_{i \in I} d_X((p_i)_-, (p_i)_+) \leq Dn.$$

Suppose that U and V are cyclic geodesics over X and conjugate in G. There are cyclic permutations $\widehat{U}'$ of $\widehat{U}$ and $\widehat{V}'$ of $\widehat{V}$ and $\widehat{C}$ such that $\widehat{C}\widehat{U}'\widehat{C}^{-1} =_G \widehat{V}'$ and

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Theorem (Antolin, Ciobanou)

Let G be hyperbolic relative to abelian groups. There is finite generating set X of G and $A \geq 0$ such that (G, X) has and A-BCD.

Analyzing the proofs one can get A-BCD, not for cyclic geodesics but for words $\mathcal{W}$ that cyclically have no trivial shortenings and contains no subword of Φ.

To solve the conjugacy problem, given two words $U^{\dagger}$, $V^{\dagger}$

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- we first obtain reduced words U and V to words in $\mathcal{W}$, $O(\ell(U^\dagger) + \ell(V^\dagger))$ steps.
- if $\max\{\ell(U), \ell(V)\} < K$ is a finite problem.
- if $\max\{\ell(U), \ell(V)\} \geq K$ we compare all possible cyclic permutations and for each of them try all conjugators of length less than K ($K \cdot \ell(U) \cdot \ell(V)$ computations).
- since the word problem is solvable in linear time, each computation takes $O(\ell(U) + \ell(V) + 2K)$.

Application to the conjugacy problem

Theorem (Antolin, Ciobanou)

Let G be hyperbolic relative to abelian groups. There is finite generating set X of G and $A \geq 0$ such that (G, X) has and A-BCD.

Analyzing the proofs one can get A-BCD, not for cyclic geodesics but for words $\mathcal{W}$ that cyclically have no trivial shortenings and contains no subword of Φ.

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Thank You!