Solvable poly-context-free groups

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Word problems of groups as languages

G a group generated by finite set X. The *word problem* of G w.r.t. X is

$$W(G,X) = \{w \in (X \cup X^{-1})^* \mid w =_G 1\}.$$

A *language* is a subset of X^* for some finite set X.

For many interesting language classes $\mathcal{C}$ (e.g. regular, context-free), $W(G,X)$ being in $\mathcal{C}$ does not depend on choice of X.

Some history

Theorem 1 (Anisimov '71)
A group has regular word problem iff it is finite.

Theorem 2 (Muller-Schupp '83, Dunwoody '85)
A f.g. group has context-free word problem iff it is virt. free.

Theorem 3 (Holt-Owens-Thomas 2008)
Let G be a f.g. group. Then G has one-counter word problem iff G is virt. cyclic, and G has word problem an intersection of finitely many one-counter languages iff G is virt. abelian.

Holt-Rees-Röver-Thomas 2005:
Groups with context-free co-word problem

Groups with poly-$\mathcal{CF}$ word problem

k-$\mathcal{CF}$: intersection of k context-free languages.

poly-$\mathcal{CF}$: k-$\mathcal{CF}$ for some $k \in \mathbb{N}$.

Facts

(i) If G is a direct product of k free groups, then G is k-$\mathcal{CF}$.
(ii) For all $k \in \mathbb{N}$, the class of k-$\mathcal{CF}$ groups is closed under taking f.g. subgroups and f.i. overgroups.

Conjecture 1 *The only k-$\mathcal{CF}$ groups are virt. f.g. subgroups of direct products of k free groups.*

Groups with multipass word problem

(Ceccherini-Silberstein, Coornaert, Fiorenzi and Schupp)

A *(deterministic) k-pass automaton* is like a (deterministic) pushdown automaton, except it reads the input k times, emptying the stack between 'passes'.

A *(deterministic) multipass language* is a language accepted by a (deterministic) k-pass automaton for some k.

Theorem 4 (C-C-F-S)
Det multipass = Boolean closure of det context-free.
Non-det multipass = poly-$\mathcal{CF}$.

Notation: $\mathcal{MG}$ = groups with det multipass WP,
$\mathcal{PG}$ = poly-$\mathcal{CF}$ groups.

Some recent evidence against the conjecture

Theorem 5 (C-C-F-S)
If G is in $\mathcal{MG}$ or $\mathcal{PG}$ and S is a subgroup of G of finite index, then the HNN extension

$$H = \langle G, t \mid tst^{-1} = s, s \in S \rangle$$

is again in the corresponding class.

It seems likely (but difficult to prove), that one can obtain groups which are not virtually f.g. subgroups of direct products of free groups by this construction.

C-C-F-S also prove closure of $\mathcal{MG}$ and $\mathcal{PG}$ under *double* with f.i. amalgamated subgroup.

The soluble case

In the case of soluble groups, the original conjecture reduces to:

Conjecture 2 *A f.g. soluble group is poly-$\mathcal{CF}$ iff it is v. ab.*

Incidentally, I also conjecture

Conjecture 3 *The word problem of a group is an intersection of finitely many $\mathcal{CF}$ languages iff it is an intersection of finitely many* deterministic $\mathcal{CF}$ *languages.*

If true, this would imply that poly-$\mathcal{CF}$ groups are a subclass of co$\mathcal{CF}$ groups.

Some evidence for the soluble case

Proposition 6 (B)
Let G be a nilpotent or polycyclic group which is not v. ab. Then G is not poly-$\mathcal{CF}$.

Proposition 7 (B)

(i) $\mathbb{Z}^k$ is not $(k-1)$-$\mathcal{CF}$.

(ii) $\mathbb{Z} \wr \mathbb{Z}$ is not poly-$\mathcal{CF}$, since $\mathbb{Z}^k \leq \mathbb{Z} \wr \mathbb{Z}$ for all $k \in \mathbb{N}$.

Proposition 8 (B)
$C_p \wr \mathbb{Z}$ is not poly-$\mathcal{CF}$ for any $p > 1$.

The groups $G(\mathbf{c})$

For $\mathbf{c} = (c_0, \ldots, c_s) \in \mathbb{Z}^{s+1}$ with $c_0, c_s \neq 0$ and $\gcd(c_0, \ldots, c_s) = 1$, define

$$G(\mathbf{c}) = \left\langle a, b \mid [b, b^{a^i}] \ (i \in \mathbb{Z}),\ b^{c_0}(b^a)^{c_1} \cdots (b^{a^s})^{c_s} \right\rangle .$$

Example: If $\mathbf{c} = (-m, 1)$, then $G(\mathbf{c}) \cong \mathrm{BS}(1, m)$.

If $G = G(\mathbf{c})$ is not v. ab., we call G a *proper Gc-group*.

Some properties of Gc-groups

Lemma Let $G = G(\mathbf{c})$ be a Gc-group with $|c_0| = |c_s| = 1$. Then G is polycyclic.

Lemma Let $G = G(\mathbf{c})$, where $\mathbf{c} = (c_0, \ldots, c_s)$ and let $\mathbf{c}' = (c_s, c_{s-1}, \ldots, c_0)$. Then $G(\mathbf{c}) \cong G(\mathbf{c}')$.

Subgroups of finitely generated soluble groups

Theorem 9 *(B - Holt 2013)*
Let G be a f.g. soluble group. Then at least one of the following holds:

(i) G *is v. ab.;*

(ii) G' *has a subgroup isomorphic to* $\mathbb{Z}^\infty$*;*

(iii) G *has a subgroup isomorphic to a proper Gc-group;*

(iv) G *has a f.g. subgroup* H *with an infinite normal torsion subgroup* U*, such that* H/U *is either free abelian or a proper Gc-group.*

Semilinear sets, bounded languages

A *linear set* has the form

$$L = \{\mathbf{c} + \sum_{i=1}^{n} \alpha_i \mathbf{p}_i \mid \alpha_i \in \mathbb{N}_0\},$$

where $\mathbf{c}, \mathbf{p}_i \in \mathbb{N}_0^r$ for some $r \in \mathbb{N}$.

A *semilinear set* is a finite union of linear sets.

The class of semilinear sets is closed under all Boolean operations.

Bounded language: $L \subseteq w_1^* \cdots w_n^*$ for some $w_1, \ldots, w_n \in X^*$.

Semilinear sets and bounded k-$\mathcal{CF}$ languages

For $L \subseteq w_1^* \cdots w_n^*$, define

$$\Phi(L) = \{(m_1, \ldots, m_n) \mid w_1^{m_1} \cdots w_n^{m_n} \in L\}.$$

Theorem 10 (Parikh '61)
If L is a bounded context-free language, then $\Phi(L)$ is a semilinear set.

Corollary 11 *If L is bounded and poly-$\mathcal{CF}$ or co$\mathcal{CF}$, then $\Phi(L)$ is a semilinear set.*

A criterion for a language to be not poly-$\mathcal{CF}$

Proposition 12 (B) *Let* $L \subseteq \mathbb{N}_0^{r+s}$ *for some* $r, s \in \mathbb{N}$. *Let* $f : \mathbb{N} \to \mathbb{N}$ *be an unbounded function s.t. for every* $k \in \mathbb{N}$, *there exists* $\mathbf{a} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$ *s.t.*

(i) $\exists \mathbf{b} \in \mathbb{N}_0^s$ *such that* $(\mathbf{a}; \mathbf{b}) \in L$.

(ii) If $(\mathbf{a}; \mathbf{b}) \in L$, *then* $\mathbf{b}(j) \geq k\sigma(\mathbf{a})$ *for all* j.

(iii) If $(\mathbf{a}; \mathbf{b}), (\mathbf{a}; \mathbf{b}') \in L$ *with* $\mathbf{b} \neq \mathbf{b}'$, *then* $|\mathbf{b}(j) - \mathbf{b}'(j)| \geq f(k)$ *for some* j.

Then L *is not a semilinear set.*

Embedding Gc-groups in $\mathbb{Q}^s \rtimes \mathbb{Z}$

Proposition 13 (B) *Let $G = G(\mathbf{c})$ be a Gc-group. Let $\{x_1, \ldots, x_s\}$ be a basis for $\mathbb{Q}^s$ over $\mathbb{Q}$, and let $\mathbb{Z} = \langle y \rangle$. Let $Q = \mathbb{Q}^s \rtimes \mathbb{Z}$, with y acting on $\mathbb{Q}^s$ by*

$$A(\mathbf{c}) = \begin{pmatrix} 0 & \ldots & 0 & -c_0/c_s \\ & & & -c_1/c_s \\ & & & \cdot \\ & I_{s-1} & & \cdot \\ & & & \cdot \\ & & & -c_{s-1}/c_s \end{pmatrix}.$$

Then $G \cong \langle x_1, y \rangle$.

Powers of the matrix $A(\mathbf{c})$

Lemma 14 (B) *Let M be a matrix of the form*

$$\begin{pmatrix} 0\ldots 0 & a_1 \\ & a_2 \\ I_{s-1} & \cdot \\ & \cdot \\ & a_s \end{pmatrix},$$

where all $a_i \in \mathbb{Q}$ and at least one $a_i \notin \mathbb{Z}$. Write $M^k = (m_{ij}^{(k)})$ for $k \in \mathbb{N}$. Then there exist $N \in \{1, \ldots, s\}$ and a prime p such that for every $k \in \mathbb{N}$ there exists some $i_k \leq ks$ with

$$-v_p(m_{Ns}^{(i_k)}) \geq k.$$

Torsion-free soluble poly-$\mathcal{CF}$ groups

Theorem 15 (B)
Let G be a proper Gc-group. Then G is neither poly-$\mathcal{CF}$ nor co$\mathcal{CF}$.

Theorem 16 (B)
Let G be a f.g. soluble poly-$\mathcal{CF}$ group. Then either

(i) G is v.ab., or (possibly)

(ii) G has a f.g. subgroup H with an infinite torsion subgroup U such that H/U is either free abelian or a proper Gc-group.

Corollary 17
A f.g. torsion-free soluble group is poly-$\mathcal{CF}$ iff it is v.ab.

Conjecture 4 *If G is a poly-$\mathcal{CF}$ group, then there is a bound on the order of finite subgroups of G.*

Possible approach: use grammars (recently developed by B and Elder) to study geometry of Cayley graph.

<u>Future plans</u>

- Is the class of poly-$\mathcal{CF}$ groups closed under taking free products? (Expect not, e.g. $\mathbb{Z}^2 * \mathbb{Z}$.)

- Groups with poly-indexed word problem

- Groups with poly-$\mathcal{CF}$ multiplication table.