Stallings graphs for quasi-convex subgroups of automatic groups

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- Efficient solutions because of automata-theoretic flavor
- We would like something similar for finitely generated subgroups of other groups

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- all vertices, except possibly for the distinguished vertex, are the origins of at least 2 edges

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- In all three cases: rely on a folding process – and we do not
- [Markus-Epstein] and [Silva, Soler-Escriva, Ventura] rely on a well-chosen set of representatives

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- and H to be *quasi-convex*.
- Note that in [Markus-Epstein] or [Silva, Soler-Escriva, Ventura], we are dealing with locally quasi-convex groups: where all finitely generated subgroups are quasi-convex

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- and automata $\mathcal{A}_a$ to compute a-multiplication for each $a \in A$: technically, an automaton on alphabet $(A \cup \{\square\})^2$, accepting all pairs of the form $(u\square^n, v\square^m)$ such that $u, v \in L$, $\mu(ua) = \mu(v)$, $|u| + n = |v| + m$ and $\min n, m = 0$

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- L is a regular set of representatives, not necessarily the set L_{geod} of geodesics (hyperbolic groups are geodesically automatic, that is, with $L = L_{\text{geod}}$)

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- These are not new results, but our construction provides a unified tool – which surely can be used for other decision problems

Definition of a Stallings graph!

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- It is with this definition in mind that we proceed with the construction

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- So, for i large enough, $(\Gamma_i, 1)$ is Stallings-like for H
- But... when are we done? How do we know when to stop the completion process?

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- For each i, when $(\Gamma_i, 1)$ is constructed, check whether it is Stallings-like, and if so, stop

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- Use the automatic structure of G: given $h = a_1 \cdots a_r \in F(A)$ and a regular subset K of L, one can construct an automaton for $L \cap \mu(Kh)$ — we pipeline the multiplyer automata for $a_1, \ldots, a_r$ to construct an h-multiplyer
- The we can check whether $L \cap \mu(Kh) \subseteq K$
- For each i, when $(\Gamma_i, 1)$ is constructed, check whether it is Stallings-like, and if so, stop
- Now we have constructed a Stallings-like graph $(\Gamma, 1)$

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- Now we have constructed a subgraph of Schreier(G, H) which contains $\Gamma_L(H)$, and which is Stallings-like
- Since (Γ_L, H) is the least rooted subgraph of the Schreier graph which is Stallings-like: we verify for each vertex whether removing it still yields a Stallings-like graph

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- [Otherwise we could decide whether a given tuple of elements generates a quasi-convex subgroup; and this problem is undecidable]

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- If this number i is part of the input, then the computation of $\Gamma_L(H)$ is exponential in i and n

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- Then H has finite index if and only if every word of L can be read in $\Gamma_L(H)$ starting from the base vertex
- This is decidable
- In that case, $\Gamma_L(H)$ is a subgraph of the (finite) Schreier graph, with all the vertices

Thank you for your attention!