Fancy divisibility in group theory

Anton A. Klyachko

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We say that two elements of a group belong to the same *tribe* if their squares are equal.

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2013 is an arbitrary positive integer; the squares (in the definition of tribes) can also be replaced by any positive integer powers.

A discouraging example

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Solutions in the symmetric group $G = S_3$.

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Thus, the total amount of solutions is $36 + 2 \cdot 9$.

This is (and must be) divisible by $|G| = 6$ but not divisible by $|G|^2$ (though the number of equations is two less than that of unknowns).

Back to high school

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The number of solutions is divisible by p if
~~there are less equations than unknowns~~. the rank of the matrix

$$\begin{pmatrix} a_1 & b_1 & \dots \\ a_2 & b_2 & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

is less than the number of unknowns.

A system of coefficient-free equations over a group G

$$\begin{cases} x^3y^3x^{-1}y[x,y]=1 \\ \\ (x,y^2)^5=1 \end{cases}$$

The *exponent-sum matrix*

$$A=\begin{pmatrix} 2 & 4 \\ 5 & 10 \end{pmatrix}$$

a_{ij} is the sum of exponents of ith unknown in jth equation.

Gordon–Rodriguez-Villegas theorem

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Theorem (Cameron Gordon & Fernando Rodriguez-Villegas, 2012)

If rank A is less than the number of unknowns, then the number of solutions is divisible by $|G|$.

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Over arbitrary group G, this corresponds to equations with coefficients.

Equations with coefficients

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A system of ~~coefficient-free~~ equations over a group $G \ni a, b, c, \dots$

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Theorem (FALSE!!!)

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$[x, a] = 1$. The exponent-sum matrix is 0 but
$|\{solutions\}| = |C(a)| < |G|$

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Theorem (K & Anna Mkrtchyan)

If rank A is less than the number of unknowns, then the number of solutions is divisible by $|C(\{a, b, \ldots\})|$.

$C(X)$ is the centraliser of a set X.

Theorem (K & Anna Mkrtchyan)

If rank(exponent-sum matrix) is less than the number of unknowns, then the number of solutions is divisible by $|C(\{a, b, \dots\})|$.

Roots of subgroups

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Exercise

If H is a subgroup of a group G, then there exists an overgroup $\widehat{G} \supseteq G, D, B$ such that, in $\widehat{G}$, $H = C(D)$ and $G = C(B)$.

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The number of elements of a group G whose cubes belong to a given subgroup H is always divisible by $|H|$.

Corollary (K & Anna Mkrtchyan)

The number of elements of a group G whose 2013th powers belong to a given subgroup H is always divisible by $|H|$.

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The number of homomorphisms $f\colon \mathbb{Z} \to G$ such that $f(2\mathbb{Z}) \subseteq H$ is divisible by $|H|$.

Roots of subgroups generalised

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Generalisation (K & Anna Mkrtchyan)

Suppose that H is a subgroup of a group G and W is a subgroup (or a subset) of a finitely generated group F with infinite abelianisation F/F'. Then the number of homomorphisms $f\colon F \to G$ such that $f(W) \subseteq H$ is always divisible by $|H|$.

First-order formulae

An arbitrary first-order formula φ over a group $G \ni a, b, \ldots$:

$$\forall z \exists t \left(z^2 y x^2 a t^{-2} x^2 y z b (xy)^5 = 1 \vee t[x, y]^2 \neq 1 \wedge (x^2 y^2 a)^3 \neq 1 \right)$$

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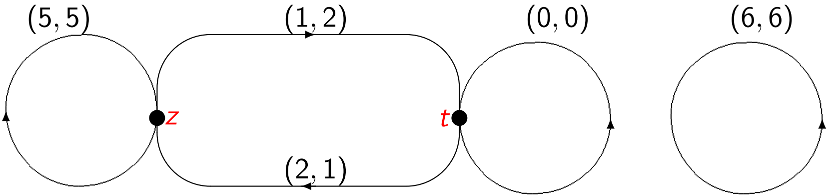
The (generalised) digraph $\Gamma(\varphi)$:



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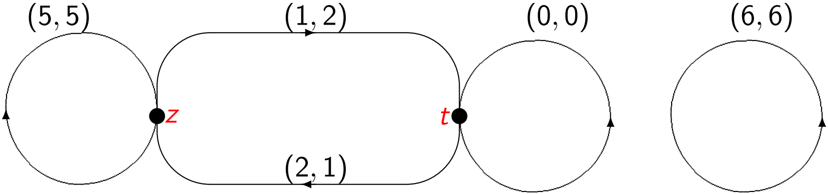

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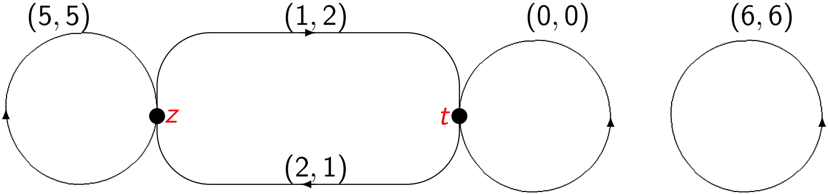

The signed sums along generating cycles:

$$(5, 5); \quad (1, 2) + (2, 1) = (3, 3); \quad (0, 0); \quad (6, 6).$$

The *exponent-sum matrix* $A(\varphi) = \begin{pmatrix} 5 & 5 \\ 3 & 3 \\ 0 & 0 \\ 6 & 6 \end{pmatrix}$

Main theorem

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Calculation-free version (K & Anna Mkrtchyan)

If

$$\#(\textit{proper occurrences of bound variables}) + \\ + \#(\textit{components of } \Gamma(\varphi)) < \#(\textit{variables}),$$

then the number of tuples of elements satisfying φ is divisible by $|C(\{a, b, \dots\})|$.

Proof.

$$\operatorname{rank}(A(\varphi)) \leqslant \#(\textit{rows}) = \dots - (\textit{the Euler characteristic of } \Gamma).$$

Theorem (K & Anna Mkrtchyan)

If $\mathrm{rank}(A(\varphi))$ is less than the number of unknowns, then the number of tuples of elements satisfying φ is divisible by $|C(\{a, b, \dots\})|$.

The order of any group divides, e.g. the following numbers:

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Conjugation theorems

A system of coefficient-free conditions (over a group $G \ni a, b, \dots$)

$$\begin{cases} x^3y^3x^{-1}y[x,y] \sim a \\ \\ (x,y^2)^5 \sim b \end{cases}$$

$$A = \begin{pmatrix} 2 & 4 \\ 5 & 10 \end{pmatrix}$$

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If rank A is less than the number of unknowns, then the number of solutions is divisible by $|G|$ (where $\sim$ stands for simultaneous conjugation).

- A. Klyachko, A. Mkrtchyan, How many tuples of group elements have a given property? arXiv:1205.2824
- A question from Mathoverflow
- Another question from Mathoverflow

Thank you!