An example of an automatic graph of intermediate growth

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Automatic groups

Automatic groups were introduced by Thurston in 1986 motivated by earlier results of Cannon.

Initial motivation was:

- understand fundamental groups of compact 3-manifolds
- make them tractable for computing

Formal Languages

X – finite alphabet
X^* – set of all finite words over X
X^∞ – set of all infinite words over X

Definition

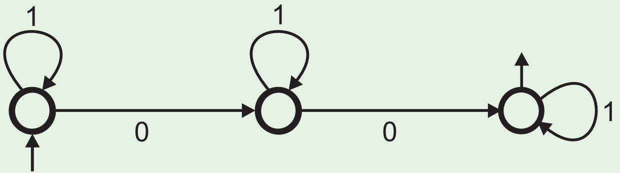
A formal language is a collection of words in X^*.

Automata – Acceptors and Regular Languages

Definition

A formal language is called regular if it is accepted by finite state automaton-acceptor.

Example



The language L accepted by this automaton is

$$\{1^n 0 1^m 0 1^k \mid n \geq 0, m \geq 0, k \geq 0\}$$

Definition (Informal)

A group $G = \langle S \rangle$ (with $S = S^{-1}$) is automatic if

- there is a regular language L over S such that $u \mapsto \overline{u}$ from L to G is onto
- right multiplication by each $s \in S \cup \{id\}$ can be performed by finite automaton

“Pros” of automatic groups

If G is automatic, then

- Word problem in G is decidable in quadratic time
- For any word $w \in S^*$ one can find its representative in L in quadratic time
- G is finitely presented
- The Dehn function of G is at most quadratic
- if G is biautomatic, then the conjugacy problem is decidable
- hyperbolic (in particular free); braid; Artin groups of finite type; Coxeter groups; most of 3-manifold groups are automatic

“Cons” of automatic groups

The following groups are NOT automatic

- infinite torsion groups
- f.g. nilpotent groups (not virtually abelian)
- some π_1(3-manifold)s
- non-abelian torsion free polycyclic groups
- $SL_n(\mathbb{Z})$
- Baumslag-Solitar groups $BS(p,q) = \langle x, y \mid y^{-1}x^p y = x^q \rangle$ unless $p = 0$, $q = 0$ or $p = \pm q$

So the class of automatic groups is NICE but NOT WIDE ENOUGH

Suggested generalizations

- Combable groups (relax requirement on the language)
- Geometric generalization of automaticity that covers all 3-manifold groups (Bridson-Gilman)
- Stackable groups (Brittenham-Hermiller)
- $\mathcal{C}$-graph automatic groups (Elder-Taback)

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We look at:

- Graph automatic groups (relax restriction on the alphabet) - Kharlampovich, Khoussainov, Miasnikov (2011)

Retains nice algorithmic properties and includes many more examples: f.g. nilpotent of class 2 and some of higher nilpotency classes; $BS(1, n)$; many metabelian and solvable groups; infinitely presented groups

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Question

Are there graph automatic groups of intermediate growth?

Let's be more specific!

$X_\diamond = X \cup \{\diamond\}$, $\diamond \notin X$ - padded alphabet.

Definition

For $(w_1, w_2) \in (X^*)^2$ a convolution $\otimes(w_1, w_2)$ is a word over $(X_\diamond)^2$ of length $\max\{|w_1|, |w_2|\}$, whose j-th symbol is (σ_1, σ_2), where

$$\sigma_i = \begin{cases} \text{the } j\text{-th symbol of } w_i, & \text{if } j \le |w_i| \\ \diamond, & \text{otherwise} \end{cases}$$

Example

$$\otimes(011, 00110) = \begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}\diamond\\1\end{pmatrix}\begin{pmatrix}\diamond\\0\end{pmatrix}$$

Regular Binary relations

Definition

Let R be a binary relation on X^*. The convolution of R is the language over $(X_\diamond)^2$ defined by

$$\otimes R = \{\otimes(w_1, w_2) \mid (w_1, w_2) \in R\} \subset (X_\diamond^2)^*$$

Definition

A binary relation R on X^* is called regular if its convolution $\otimes R$ is a regular language over $(X_\diamond)^2$.

Automatic vs. Graph Automatic groups

Definition (Automatic (Thurston))

A f.g. group $G = \langle S \rangle$ is called automatic if

- There exists a regular language $L \subset S^*$ such that $\overline{} : L \to G$ is onto
- The relations $E_s = \{(u, v) \mid u, v \in L, \overline{u} = \overline{v}s\}$ on S^* are regular for $s \in S \cup \{id\}$

Definition (Graph Automatic (KKM))

A f.g. group $G = \langle S \rangle$ is called Graph automatic if there is a finite alphabet X such that

- There exists a regular language $L \subset X^*$ and an onto map $\overline{} : L \to G$
- The relations $E_s = \{(u, v) \mid u, v \in L, \overline{u} = \overline{v}s\}$ on X^* are regular for $s \in S \cup \{id\}$

X need not coincide with a generating set S.

More general definition of graph automaticity

Let $\Gamma = (V, E, \sigma\colon E \to S)$ be a labeled graph.
We interpreted it as a system of $|S|$ binary relations E_s on V:

$$E_s = \{(v, v') \mid (v, v') \in E \text{ and the label of } (v, v') \text{ is } s\}.$$

Each map $\overline{}\colon V \to X^*$ induces $|S|$ binary relations $\overline{E}_s$ on X^*

$$\overline{E}_s = \{(\overline{v}, \overline{v'}) \mid (v, v') \in E_s\}.$$

Definition

$\Gamma = (V, E, \sigma\colon E \to S)$ is called automatic, if there is a finite alphabet X and an injective map $\overline{}\colon V \to X^*$ such that

- $\overline{V}$ is a regular language over X and
- $\overline{E}_s$ is a regular binary relation on X^* for each $s \in S$.

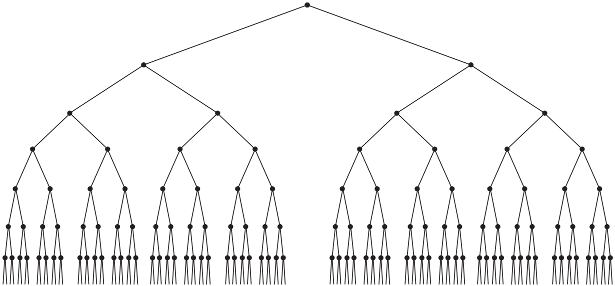
More general definition of graph automaticity

Proposition

A f.g. group $G = \langle S \rangle$ is graph automatic $\Leftrightarrow$ Cayley graph $\mathrm{Cay}(G, S)$ with respect to S is automatic.

Automata – transducers

$V(T) = X^*, \qquad X = \{0, \ldots, d-1\}$ – alphabet

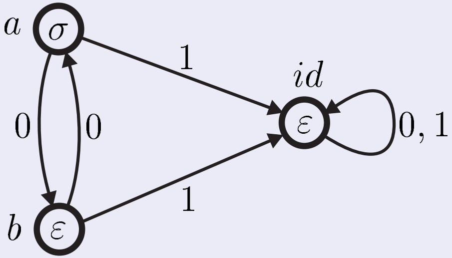


$$G < \mathrm{Aut}\, T$$

Action on T given by finite initial automaton

Definition (By Example)

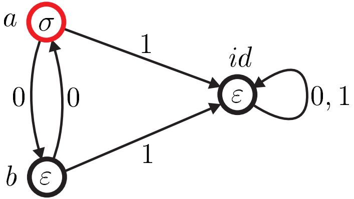
$S_2 = \{\varepsilon, \sigma\}$ acts on $X = \{0, 1\}$.



$\mathcal{A}$ — noninitial automaton,
$\mathcal{A}_q$ — initial automaton, $q \in \{a, b, id\}$.

$\mathcal{A}_q$ acts on X^* (and on T)

Input:	0	0	0	0	1	0	1	1
	$\downarrow$							
States:	a	b	a	b	a	id	id	id
	$\downarrow$							
Output:	1	0	1	0	0	0	1	1



Input:	0	0	0	0	1	0	1	1
States:	a	b	a	b	a	id	id	id
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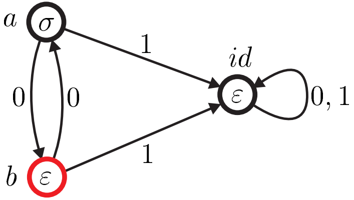

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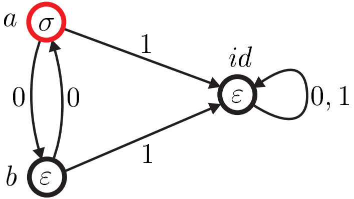

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				↓				
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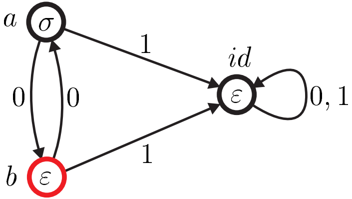

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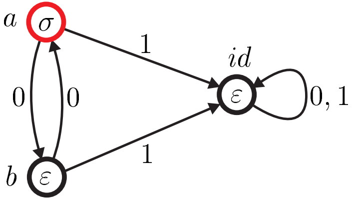

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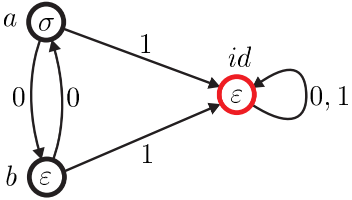

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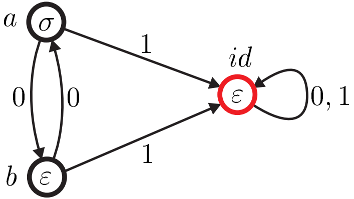

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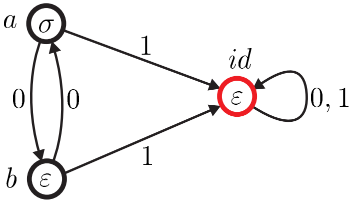

Definition of automaton group

Given an automaton A every state q defines an automorphism A_q of X^*

Definition

The automaton group generated by automaton A is a group

$$G(A) = \langle A_q \mid q \text{ is a state of A} \rangle < \mathrm{Aut}\, X^*$$

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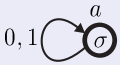
Example



$a(w) = \overline{w}$. Thus $a^2 = 1$ and $G(A) \simeq C_2$.

Automata groups as a source of counterexamples

- Burnside problem on infinite periodic groups
- Milnor problem on groups of intermediate growth
- Day problem on amenability
- Atiyah conjecture on L^2 Betti numbers

Schreier graphs

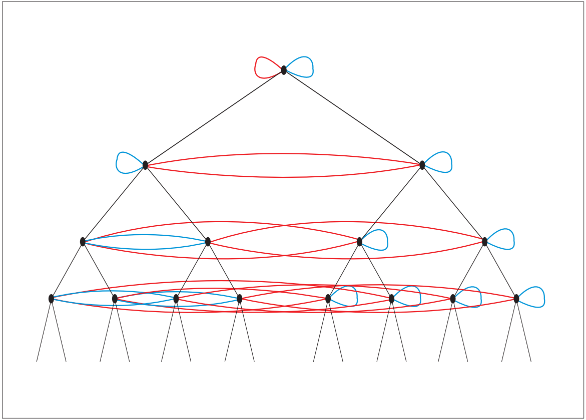
Let $G = \langle S \rangle$ act transitively on X.

Definition

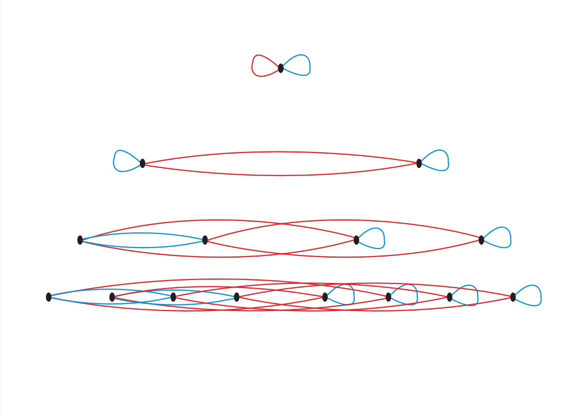
The Schreier graph $\Gamma(G, X, S)$ of the action of G on X with respect to generating set S is the graph with set of vertices X and edges

$$x \xrightarrow{s} x^s$$

Schreier Graphs



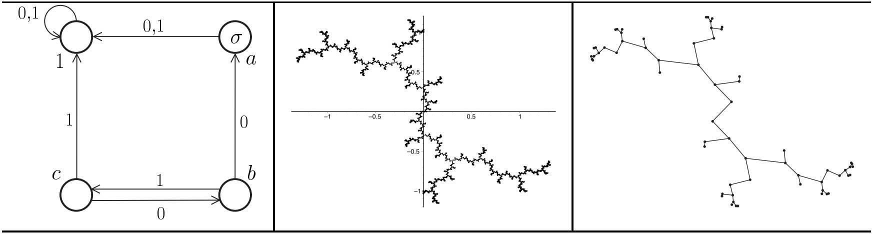
Schreier Graphs



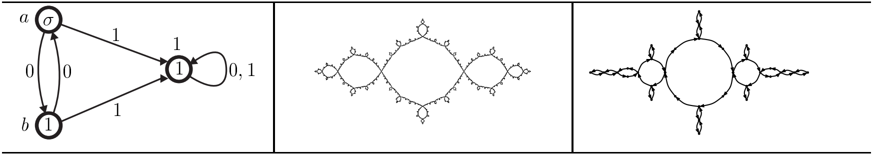
Why Schreier graphs?

- Are usually simpler than Cayley graphs
- Describe the action at the level of orbits
- If Schreier graph of G is non-amenable, then G is non-amenable.
- Are used to construct expanders
- Connect groups acting on rooted trees and holomorphic dynamics

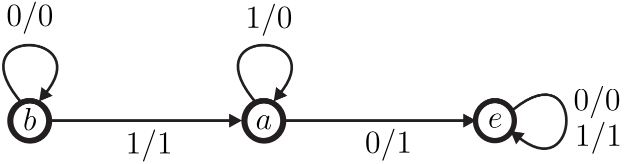
$IMG(z^2 + i)$



$IMG(z^2 - 1)$



Automaton generating group G

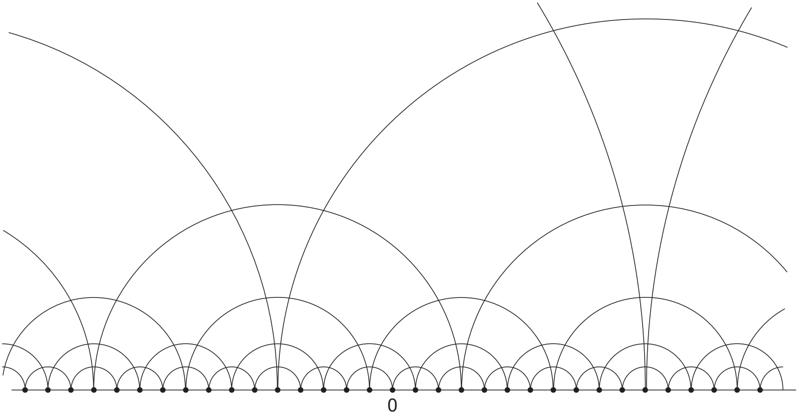


Theorem (Bondarenko, Ceccherini-Silberstein, Donno, Nekrashevych, 2012)

All Schreier graphs Γ_ω for $\omega \in \{0,1\}^\infty$ of the group G have intermediate growth. More specifically, the growth function satisfies

$$n^{\frac{1}{2}\log_2 n} \preceq |B(\omega, n)| \preceq n^{\log_2 n}$$

Graph $\Gamma_{(01)^\infty}$



Theorem (Miasnikov,S.)

The graph $\Gamma_{(01)^\infty}$ is an automatic graph of intermediate growth.

Definition of $\overline{}$

Definition

$\omega = x_1x_2x_3\ldots$ and $\omega' = y_1y_2y_3\ldots$ in X^∞ are called cofinal if there exists $N > 0$ such that $x_n = y_n$ for all $n \geq N$.

Proposition (Bondarenko, Ceccherini-Silberstein, Donno, Nekrashevych, 2012)

The orbit of $\omega = (01)^\infty$ coincides with a cofinality class of $(01)^\infty$.

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Thus, each vertex of $\Gamma_{(01)^\infty}$ is labelled by an infinite word over X that is cofinal with $(01)^\infty$.

Definition of $\overline{}$

For

$$\begin{array}{lcccccccccc} \omega & = & x_1 & x_2 & x_3 & \ldots & x_k & 0 & 1 & 0 & 1\ldots \\ (01)^\infty & = & 0 & 1 & 0 & \ldots & 1 & 0 & 1 & 0 & 1\ldots \end{array}$$

where $x_k \neq 1$, define

$$\overline{\omega} = x_1x_2x_3\ldots x_k$$

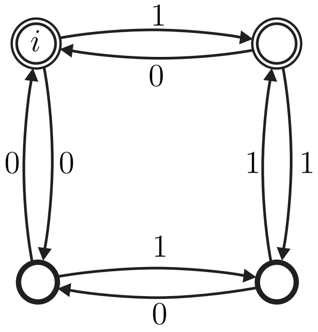
Example

- $\overline{(01)^\infty} = \emptyset$
- $\overline{110011(01)^\infty} = 11001$

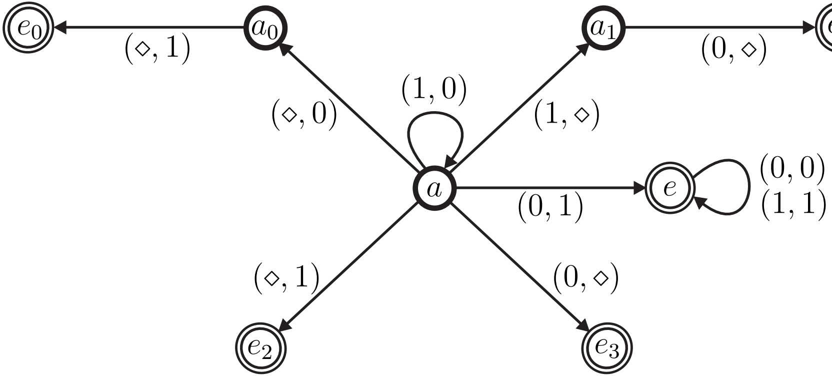
Automaton $\mathcal{A}_V$ accepting $\overline{V(\Gamma_{(01)^\infty})}$

Observation

$\overline{V(\Gamma_{(01)^\infty})}$ consists of the empty word and words whose last letter is different from corresponding letter of $(01)^\infty$.



Automaton $\mathcal{A}_a$ accepting L_a



Automaton $\mathcal{A}_b$ accepting L_b