An example of an automatic graph of intermediate growth

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Automatic groups

Automatic groups were introduced by Thurston in 1986 motivated by earlier results of Cannon.

Initial motivation was:

- understand fundamental groups of compact 3-manifolds
- make them tractable for computing
Formal Languages

$X$ – finite alphabet
$X^* – set of all finite words over $X$
$X^\infty – set of all infinite words over $X$

**Definition**

A *formal language* is a collection of words in $X^*$. 
Definition

A formal language is called **regular** if it is accepted by finite state automaton-acceptor.

Example

The language $L$ accepted by this automaton is

$$\{1^n01^m01^k \mid n \geq 0, m \geq 0, k \geq 0\}$$
Definition (Informal)

A group $G = \langle S \rangle$ (with $S = S^{-1}$) is automatic if

- there is a regular language $L$ over $S$ such that $u \mapsto \overline{u}$ from $L$ to $G$ is onto
- right multiplication by each $s \in S \cup \{id\}$ can be performed by finite automaton
“Pros” of automatic groups

If $G$ is automatic, then

- Word problem in $G$ is decidable in quadratic time
- For any word $w \in S^*$ one can find its representative in $L$ in quadratic time
- $G$ is finitely presented
- The Dehn function of $G$ is at most quadratic
- If $G$ is biautomatic, then the conjugacy problem is decidable
- hyperbolic (in particular free); braid; Artin groups of finite type; Coxeter groups; most of 3-manifold groups are automatic
“Cons” of automatic groups

The following groups are NOT automatic

- infinite torsion groups
- f.g. nilpotent groups (not virtually abelian)
- some $\pi_1(3$-manifold)s
- non-abelian torsion free polycyclic groups
- $SL_n(\mathbb{Z})$
- Baumslag-Solitar groups $BS(p, q) = \langle x, y \mid y^{-1}x^py = x^q \rangle$ unless $p = 0$, $q = 0$ or $p = \pm q$

So the class of automatic groups is NICE but NOT WIDE ENOUGH
Suggested generalizations

- Combable groups (relax requirement on the language)
- Geometric generalization of automaticity that covers all 3-manifold groups (Bridson-Gilman)
- Stackable groups (Brittenham-Hermiller)
- $C$-graph automatic groups (Elder-Taback)
Suggested generalizations

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We look at:

- Graph automatic groups (relax restriction on the alphabet) - Kharlampovich, Khoussainov, Miasnikov (2011)

Retains nice algorithmic properties and includes many more examples: f.g. nilpotent of class 2 and some of higher nilpotency classes; $BS(1, n)$; many metabelian and solvable groups; infinitely presented groups
Suggested generalizations

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Question

Are there graph automatic groups of intermediate growth?
Let’s be more specific!

\[ X_{\diamond} = X \cup \{\diamond\}, \diamond \not\in X \] - padded alphabet.

**Definition**

For \((w_1, w_2) \in (X^*)^2\) a convolution \(\otimes(w_1, w_2)\) is a word over \((X_{\diamond})^2\) of length \(\max\{|w_1|, |w_2|\}\), whose \(j\)-th symbol is \((\sigma_1, \sigma_2)\), where

\[
\sigma_i = \begin{cases} 
\text{the } j\text{-th symbol of } w_i, & \text{if } j \leq |w_i| \\
\diamond, & \text{otherwise}
\end{cases}
\]

**Example**

\[
\otimes(011, 00110) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \diamond \\ \diamond \end{pmatrix}
\]
Regular Binary relations

Definition
Let $R$ be a binary relation on $X^*$. The convolution of $R$ is the language over $(X^*)^2$ defined by

$$\otimes R = \{\otimes (w_1, w_2) \mid (w_1, w_2) \in R\} \subset (X^*)^2$$

Definition
A binary relation $R$ on $X^*$ is called regular if its convolution $\otimes R$ is a regular language over $(X^*)^2$. 
Automatic vs. Graph Automatic groups

Definition (Automatic (Thurston))

A f.g. group $G = \langle S \rangle$ is called **automatic** if

- There exists a regular language $L \subset S^*$ such that $\bar{\rightarrow}: L \to G$ is onto
- The relations $E_s = \{ (u, v) \mid u, v \in L, \bar{u} = \bar{v}s \}$ on $S^*$ are regular for $s \in S \cup \{ id \}$

Definition (Graph Automatic (KKM))

A f.g. group $G = \langle S \rangle$ is called **Graph automatic** if there is a finite alphabet $X$ such that

- There exists a regular language $L \subset X^*$ and an onto map $\bar{\rightarrow}: L \to G$
- The relations $E_s = \{ (u, v) \mid u, v \in L, \bar{u} = \bar{v}s \}$ on $X^*$ are regular for $s \in S \cup \{ id \}$

$X$ need not coincide with a generating set $S$. 
More general definition of graph automaticity

Let \( \Gamma = (V, E, \sigma : E \rightarrow S) \) be a labeled graph. We interpreted it as a system of \(|S|\) binary relations \( E_s \) on \( V \):

\[
E_s = \{ (v, v') \mid (v, v') \in E \text{ and the label of } (v, v') \text{ is } s \}.
\]

Each map \( \overline{-} : V \rightarrow X^* \) induces \(|S|\) binary relations \( \overline{E}_s \) on \( X^* \):

\[
\overline{E}_s = \{ (\overline{v}, \overline{v}') \mid (v, v') \in E_s \}.
\]

**Definition**

\( \Gamma = (V, E, \sigma : E \rightarrow S) \) is called **automatic**, if there is a finite alphabet \( X \) and an injective map \( \overline{-} : V \rightarrow X^* \) such that

- \( \overline{V} \) is a regular language over \( X \) and
- \( \overline{E}_s \) is a regular binary relation on \( X^* \) for each \( s \in S \).
More general definition of graph automaticity

Proposition

A f.g. group $G = \langle S \rangle$ is graph automatic $\iff$ Cayley graph $Cay(G, S)$ with respect to $S$ is automatic.
$V(T) = X^*$, \hspace{1cm} X = \{0, \ldots, d - 1\} \text{ -- alphabet}$

$G < \text{Aut } T$
Action on $T$ given by finite initial automaton

**Definition (By Example)**

$S_2 = \{\varepsilon, \sigma\}$ acts on $X = \{0, 1\}$.

\[ \mathcal{A} \quad \text{noninitial automaton}, \]
\[ \mathcal{A}_q \quad \text{initial automaton, } q \in \{a, b, id\}. \]

$A_q$ acts on $X^*$ (and on $T$)
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<th>Input:</th>
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<td>0 0 0 0 1 0 1 1</td>
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<th>States:</th>
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<td>a b a b a id id id</td>
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![Diagram](image-url)
Input: 0 0 0 0 1 0 1 1 1
States: \(a\) \(b\) \(a\) \(b\) \(a\) \(id\) \(id\) \(id\)
Output: 1 0 1 0 0 0 1 1 1
Input: 0 0 0 0 1 0 1 1 1

States: a b a b a id id id

Output: 1 0 1 0 0 0 1 1 1

- Graph with states and transitions:
  - States: ε, σ
  - Transitions:
    - a: ε → σ, σ → 1
    - b: ε → ε
    - id: ε → id, id → ε

- Input sequence:
  - 0 0 0 0 1 0 1 1 1

- Output sequence:
  - 1 0 1 0 0 0 1 1 1
Input: 0 0 0 0 1 0 1 1 1

States:

\[ a \ b \ a \ b \ a \ id \ id \ id \]

Output: 1 0 1 0 0 0 1 1 1

\[ a \sigma \]

\[ b \varepsilon \]

\[ 0 \ 0 \ 1 \]

\[ \varepsilon \]

\[ 0, 1 \]
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![Graph Diagram](image-url)
Input: \[\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
\end{array}\]

States: \[\begin{array}{cccccccccc}
a & b & a & b & a & \text{id} & \text{id} & \text{id} \\
\end{array}\]

Output: \[\begin{array}{cccccccccc}
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
\end{array}\]
**Input:**

```
0 0 0 0 1 0 1 1
```

**States:**

```
a b a b a id id id
```

**Output:**

```
1 0 1 0 0 0 1 1
```

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**Diagram:**

- **Input Symbol: A**
- **Output Symbol: B**
- **State Symbol: C**
- **Transition Symbols: D, E**

**Vertices:**
- **Start State (A):** Input starts here.
- **End State (C):** Output ends here.
- **Intermediate States:**
  - **B:** (State): Transition to output.
  - **D:** (State): Transition to output.

**Edges:**
- **Transition Arrows:**
  - From **A** to **B** with symbol **D**.
  - From **B** to **C** with symbol **E**.
  - From **C** to **A** with symbol **E**.

**Markings:**
- **Start State (A):** **B**
- **End State (C):** **C**
Input:

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States:

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Output:

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Definition of automaton group

Given an automaton $A$ every state $q$ defines an automorphism $A_q$ of $X^*$

**Definition**

The **automaton** group generated by automaton $A$ is a group

$$G(A) = \langle A_q \mid q \text{ is a state of } A \rangle < Aut X^*$$
Definition of automaton group

Given an automaton \( A \) every state \( q \) defines an automorphism \( A_q \) of \( X^* \)

\[
G(A) = \langle A_q \mid q \text{ is a state of } A \rangle < \text{Aut } X^*
\]

Example

\[
0, 1 \xrightarrow{a} \sigma
\]

\( a(w) = \overline{w} \). Thus \( a^2 = 1 \) and \( G(A) \simeq C_2 \).
Automata groups as a source of counterexamples

- Burnside problem on infinite periodic groups
- Milnor problem on groups of intermediate growth
- Day problem on amenability
- Atiyah conjecture on $L^2$ Betti numbers
Let $G = \langle S \rangle$ act transitively on $X$.

**Definition**

The **Schreier graph** $\Gamma(G, X, S)$ of the action of $G$ on $X$ with respect to generating set $S$ is the graph with set of vertices $X$ and edges

![Diagram of Schreier graph](image)
Schreier Graphs
Schreier Graphs
Why Schreier graphs?

- Are usually simpler than Cayley graphs
- Describe the action at the level of orbits
- If Schreier graph of $G$ is non-amenable, then $G$ is non-amenable.
- Are used to construct expanders
- Connect groups acting on rooted trees and holomorphic dynamics
\[ \text{IMG}(z^2 + i) \]

\[ \text{IMG}(z^2 - 1) \]
Theorem (Bondarenko, Ceccherini-Silberstein, Donno, Nekrashevych, 2012)

All Schreier graphs $\Gamma_\omega$ for $\omega \in \{0, 1\}^\infty$ of the group $G$ have intermediate growth. More specifically, the growth function satisfies

$$n^{\frac{1}{2}\log_2 n} \leq |B(\omega, n)| \leq n^{\log_2 n}$$
Theorem (Miasnikov, S.)

The graph $\Gamma_{(01)^\infty}$ is an automatic graph of intermediate growth.
Definition of $\omega = x_1 x_2 x_3 \ldots$ and $\omega' = y_1 y_2 y_3 \ldots$ in $X^\infty$ are called cofinal if there exists $N > 0$ such that $x_n = y_n$ for all $n \geq N$.

Proposition (Bondarenko, Ceccherini-Silberstein, Donno, Nekrashevych, 2012)

The orbit of $\omega = (01)^\infty$ coincides with a cofinality class of $(01)^\infty$. 
Definition

\( \omega = x_1 x_2 x_3 \ldots \) and \( \omega' = y_1 y_2 y_3 \ldots \) in \( X^\infty \) are called **cofinal** if there exists \( N > 0 \) such that \( x_n = y_n \) for all \( n \geq N \).

Proposition (Bondarenko, Ceccherini-Silberstein, Donno, Nekrashevych, 2012)

*The orbit of \( \omega = (01)^\infty \) coincides with a cofinality class of \( (01)^\infty \).*

Thus, each vertex of \( \Gamma_{(01)^\infty} \) is labelled by an infinite word over \( X \) that is cofinal with \( (01)^\infty \).
Definition of

For

\[ \omega = x_1 x_2 x_3 \ldots x_k 0 1 0 1 \ldots \]
\[ (01)_{\infty} = 0 1 0 \ldots 1 0 1 0 1 \ldots \]

where \( x_k \neq 1 \), define

\[ \overline{\omega} = x_1 x_2 x_3 \ldots x_k \]

Example

- \( (01)_{\infty} = \emptyset \)
- \( 110011(01)_{\infty} = 11001 \)
Automaton $A_V$ accepting $V(\Gamma_{(01)\infty})$

**Observation**

$V(\Gamma_{(01)\infty})$ consists of the empty word and words whose last letter is different from corresponding letter of $(01)\infty$. 
Automaton $\mathcal{A}_a$ accepting $L_a$
Automaton $\mathcal{A}_b$ accepting $L_b$