Quadratic equations in free monoids with involution and surface train tracks

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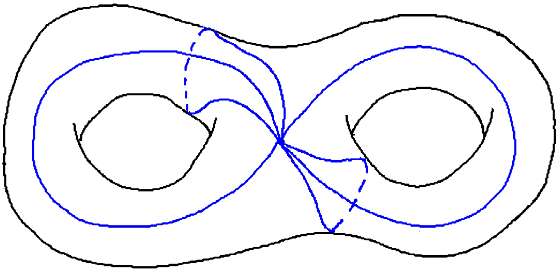
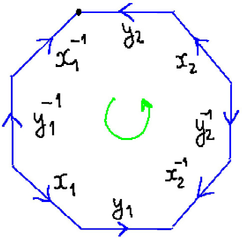
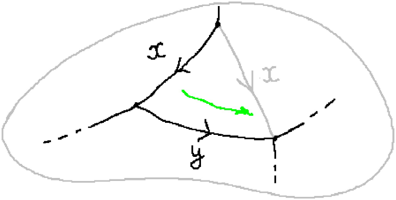




$$[x_1, y_1][x_2, y_2]$$

Fix a set X of generators. A *quadratic* word is an element $W \in F_X$ such that each letter occurring in W occurs twice. Every quadratic word W represents a compact surface S_W.



$$W = \dots xy \dots$$

$$x \mapsto xy^{-1}$$

Every quadratic word is equivalent to a word of the form

$$[x_1, y_1][x_2, y_2] \dots [x_g, y_g] \quad \text{or} \quad x_1^2 x_2^2 \dots x_g^2$$

'Equivalent' means 'lies in the same orbit under the action of $\mathrm{Aut}(F_X)$'.

This is helpful when solving quadratic equations in a free group F_A.

Introducing coefficients may be interpreted as passing to compact surfaces with boundary.

Quadratic equations $W = 1$ in F_A can have two extra forms up to equivalence:

$$[x_1, y_1][x_2, y_2] \dots [x_g, y_g]\, c_0\, z_1^{-1} c_1 z_1 \ \dots\ z_m^{-1} c_m z_m$$

$$x_1^2 x_2^2 \dots x_g^2\, c_0\, z_1^{-1} c_1 z_1 \ \dots\ z_m^{-1} c_m z_m$$

where $c_0, c_1, \dots c_m \in F_A$ and x_i, y_i are variables.

Take free monoids with involution instead of free groups.

Initial motivation:

- Getting an upper bound on the shortest solution of a quadratic equation if a free monoid with involution.
- Understanding "the quadratic part" of the elimination process for equations in free groups.

A *monoid with involution* is a monoid with an extra involution $x \mapsto x^{-1}$ satisfying the identity

$$(xy)^{-1} = y^{-1}x^{-1}$$

The free monoid with involution on a generating set A is the set of words in the doubled alphabet

$$A^{\pm 1} = A \cup \{a^{-1} \mid a \in A\}$$

with operations naturally defined.

Instead of quadratic equations $W = 1$, we have to consider *quadratic systems* of equations:

$$L_i = R_i, \quad i = 1, 2, \ldots, k. \tag{1}$$

Here $L_i, R_i \in M_{A \cup X}$ (the free monoid with involution freely generated by $A \cup X$) and each variable occurring in the system, occurs exactly twice.

A system (1) is *orientable* if the surface produced by the formal system $\{L_i R_i^{-1} = 1\}$ is orientable.

Theorem

If an orientable quadratic system S of equations $\{L_i = R_i,\ i \in I\}$ in a free monoid with involution M_A is solvable then it has a solution of size

$$\leq O(c\ \exp(n^6))$$

where c is the total length of coefficients of S and n is the number of variables in S.

Corollary

Let M be a free monoid (with or without involution) with at least two free generators. Then the Diophantine problem for orientable quadratic systems in M is NP-complete.

Theorem (description of solutions, an easy form)

Let S be a quadratic system of equations in a free monoid with involution M_A. Then there exists and can be effectively computed from S, a finite set of sequences of the following form producing all solutions of S:

$$\begin{tikzcd}
M_0 \arrow[loop above, "U_0"] \arrow[r, "\phi_1"] & M_1 \arrow[loop above, "U_1"] \arrow[r, "\phi_2"] & \dots \arrow[r, "\phi_{r-1}"] & M_{r-1} \arrow[loop above, "U_{r-1}"] \arrow[r, "\phi_r"] & M_r
\end{tikzcd}$$

where

1. *$M_i = N_{S_i}$ for certain quadratic systems S_i and ϕ_i is an M_A-homomorphism; $S_0 = S$ and the last system S_r is trivial, that is, $M_r = M_A * M_V$ for some finite V;*
2. *U_i is a monoid of injective M_A-endomorphisms, effectively given as a regular language on a finite set of M_A-endomorphisms.*

Theorem (description of solutions, a difficult form)

Let S be an orientable quadratic system of equations in a free monoid with involution M_A. Then there exists and can be effectively computed from S, a finite set of sequences of the following form producing all solutions of S:

$$\begin{tikzcd} M_0 \arrow[r, bend left, "V_0"] & M_0' \arrow[r, "\phi_1"] & M_1 \arrow[r, bend left, "V_1"] & M_1' \arrow[r, "\phi_2"] & \ldots \arrow[r, "\phi_{r-1}"] & M_{r-1} \arrow[r, bend left, "V_{r-1}"] & M_{r-1}' \arrow[r, "\phi_r"] & M_r \end{tikzcd}$$

where

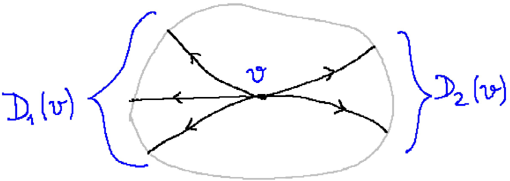
1. *for each i, $M_i = M_{S_i}$ and $M_i' = M_{S_i'}$ for certain quadratic systems S_i and S_i'; each ϕ_i is a M_A-homomorphism; $S_0 = S$ and the last system S_r is trivial.*
2. *S_i and S_i' are combinatorially equivalent for all $i = 0, \ldots, r-1$;*
3. *The length r of each sequence is bounded by $O(n^3)$ where n is the number of variables in S.*
4. *V_i is a regular set of injective homomorphisms which can be defined by a finite graph of homomorphisms.*

A *surface train track* consists of:

- a finite graph T embedded (as a topological graph) in a closed compact surface S;
- a partition of the star $St(v)$ of each vertex v into two nonempty subsets $D_1(v)$ and $D_2(v)$ called *directions*; they form two continuous subsets of $St(v)$ under the cyclic ordering induced by the embedding $T \to S$.

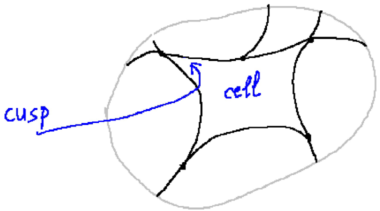
(The star of a vertex v is the set of all directed edges coming out of v.)

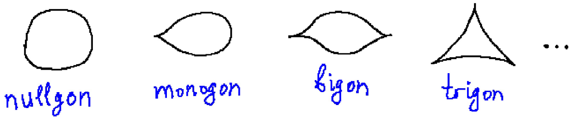
Usually it is assumed that T is *geometrically* embedded in S: edges of T are smooth arcs in S and the edges in $D_1(v)$ and $D_2(v)$ are coming out of v in two opposite directions.



Always assume:

- T is connected;
- each component of $S - T$ is simply connected; it is a *cell*.



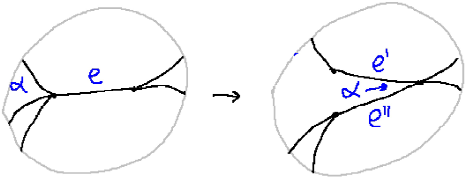


Elementary transformations:

(T1): removing and introducing bivalent vertices

(T2): elementary unzipping operations

Unzipping of an edge e at a cusp α:

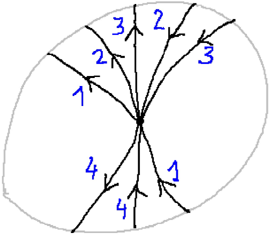


Definition

Two surface train tracks T_1 and T_2 are *combinatorially equivalent* if each one is obtained from the other by a sequence of transformations (T1) and (T2)

(I call the RST closure of (T1) & (T2) *weak combinatorial equivalence*)

Representing surface train tracks by patterns:



$$\longrightarrow \quad \begin{bmatrix} 1\,2\,3\,2^{-1}\,3^{-1} \\ 4^{-1}\,4\,1 \end{bmatrix}$$

A *pattern* is a finite set of pairs $\{(U_i, V_i)\}$ of nonempty words $U_i, V_i \in M_X$ such that each variable $x \in X$ occurs in the words U_i and V_i totally twice. (So a pattern may be viewed as a formal coefficient-free quadratic system of equations with variables in M_X; ‘formal’ means here that we do not consider solutions of such systems.)

A special class of surface train tracks:

Assume that train tracks are

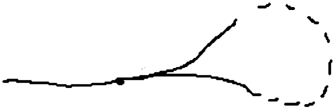
- single vertex;
- non-splittable;
- orientable.

A pattern $\begin{bmatrix} U \\ V \end{bmatrix}$ is *balanced* if each letter occurs once in U and once in V.

A pattern $\begin{bmatrix} U \\ V \end{bmatrix}$ represents a *splittable* train track if there is a nontrivial partition $U = U_1V_2$ and $V = V_1V_2$ such that U_1V_1 and U_2V_2 have disjoint sets of variables and at least of the patterns $\begin{bmatrix} U_i \\ V_i \end{bmatrix}$ is balanced.

A train track T is *reversible* if there is a train path in T which passes an edge twice in opposite directions. '*Irreversible*' is the negation.

Reversible = U-turnable:



Easy observation: A pattern $P = \begin{bmatrix} U \\ V \end{bmatrix}$ is balanced iff the train track $\mathcal{T}(P)$ represented by P is irreversible.

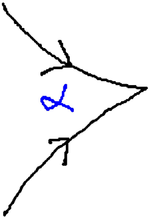
A classification of (single vertex non-splittable orientable) train tracks up to equivalence

Train tracks are *marked*: assumed to have a fixed marking of their cusps. Elementary transformations and equivalence are applied to marked train tracks.

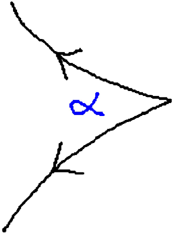
Definition

The *combinatorial type* of a marked surface train track T consists of the following data:

- Type of T: reversible or irreversible.
- The partition of the set of cusps of T into a collection of cyclically ordered sets which occur in the boundary loops of cells of T.
- If T is irreversible, we add an extra information: the partition of the set of cusps into two subsets C^+ and C^- induced by the partition $E = E^+ \cup E^-$ of the directed edges with respect to the distinguished direction.



$\alpha \in C^+$



$\alpha \in C^-$

Irreversible train tracks:

- Eliminating bigons
- The family of rigid train tracks
- Exceptional cases
- The general case

Proposition

Let T and T' be two irreversible train tracks of the same combinatorial type and let $\hat{T}$ and $\hat{T}'$ be train tracks obtained from T and T' by elimination of the same bigon. Then T and T' are equivalent if and only if $\hat{T}$ and $\hat{T}'$ are equivalent.

The family of rigid irreversible train tracks

Let IR_n be the following family of patterns:

$$IR_n = \begin{bmatrix} 1\,2\,\ldots\,n \\ n\,\ldots\,2\,1 \end{bmatrix}, \quad n \geq 1.$$

The train track $\mathcal{T}(IR_n)$ has a single $(2n-2)$-gon if n is even and two $(n-1)$-gons if n is odd. We call a train track *rigid* if it is equivalent to $\mathcal{T}(IR_n)$.

Proposition

Let $k \geq 1$.

For each combinatorial type of irreversible train tracks of signature $\{4k-2\}$ there is a single class of rigid train tracks.

For each combinatorial type of irreversible train tracks of signature $\{2k, 2k\}$ there are k classes of rigid train tracks which are distinguished by position of one $2k$-gon with respect to the other.

Proposition

Any irreversible train track with no eliminable bigons (i.e. either it has no bigons or has a bigon as a single cell) of genus ≤ 2 *is rigid. They have signatures* $\{0, 0\}$, $\{2\}$, $\{6\}$ *and* $\{4, 4\}$.

For each combinatorial type of signature $\{10\}$ *(which has genus 3) there is precisely one equivalence class of non-rigid irreversible train tracks. It is represented by a pattern*

$$\begin{bmatrix} 1\,2\,3\,4\,5\,6 \\ 6\,2\,3\,4\,5\,1 \end{bmatrix}$$

The cases above cover all the possibilities for irreversible train tracks without bigons with at most 6 *edges.*

Proposition

For each combinatorial type of irreversible train tracks without bigons with at least 7 *edges there are two equivalence classes of non-rigid train tracks distinguished by a certain invariant called the* parity.

These combinatorial types cover:

- *irreversible train tracks of genus* 3 *of signature* $\{4, 8\}$*;*
- *irreversible train tracks without bigons of genera* ≥ 4*.*