Some results about growth and cogrowth for finitely generated groups

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Part 1 joint work with

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Part 2 joint work with

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Part 1: cogrowth

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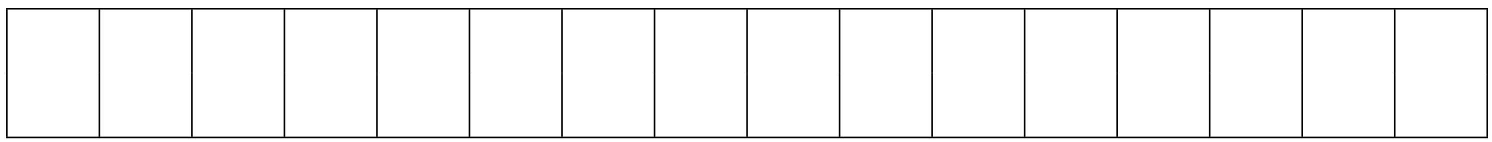
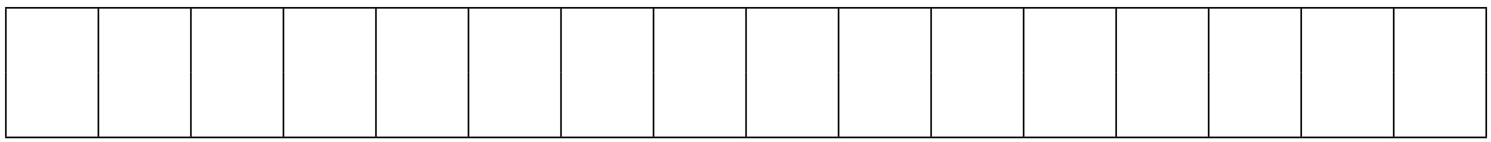
$\rho = \limsup_{n \to \infty} c(n)^{1/n}$ is the **cogrowth rate** for (G,X)

and $\sum_{n=0}^{\infty} c(n) z^n$ is the **cogrowth series** for (G,X)

Eg: $\mathbb{Z} = \langle a \mid - \rangle$



Eg: $\mathbb{Z} = \langle a \mid - \rangle$



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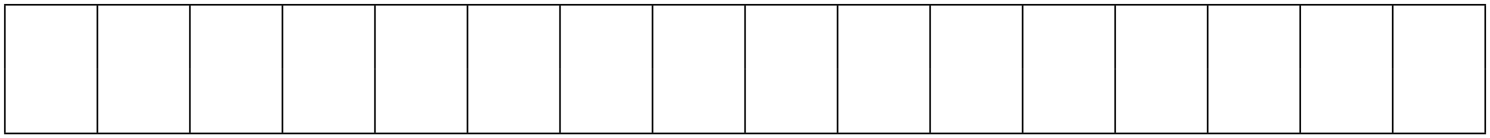


so $c(2n+1) = 0$ and $c(2n) = \binom{2n}{n}$ which is approx $4^n = 2^{2n}$

so $\rho = 2$

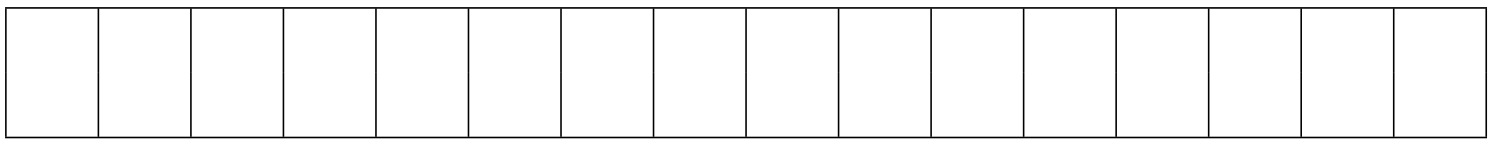
(Note: the number of words on X is $|X|^n$ so $\rho \leq |X|$)

Eg: $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$

How many return paths of length n (even) are there in the grid?

rotate $\pi/4$, four types of edges: 1 2 3 4

Eg: $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$



Choose $n/2$ boxes to be $\uparrow$ vertical, and (independently) $n/2$ boxes to be $\uparrow$ horizontal.

So $c(n) = \binom{n}{n/2}\binom{n}{n/2}$.

Eg: $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$

$\binom{n}{n/2}\binom{n}{n/2}$ grows like $\dfrac{4^{n+1/2}}{\pi n}$ so $\rho = 4$.

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Note: the number of words on X is $|X|^n$ so $\rho \leq |X|$.

Note also: the n in the denominator means that the cogrowth series for $\mathbb{Z}^2$ is not algebraic.

Eg: $\mathsf{F}_2 = \langle a, b \mid \ \rangle$

Recall (Muller-Schupp) that the **word problem** for F_2 is context-free

$\mathsf{S} \to a\mathsf{A}a^{-1}\mathsf{S} \mid b\mathsf{B}b^{-1}\mathsf{S} \mid a^{-1}\mathsf{C}a\mathsf{S} \mid b^{-1}\mathsf{D}b\mathsf{S} \mid \epsilon$

$\mathsf{A} \to a\mathsf{A}a^{-1}\mathsf{A} \mid b\mathsf{B}b^{-1}\mathsf{A} \mid b^{-1}\mathsf{D}b\mathsf{A} \mid \epsilon$

$\mathsf{B} \to a\mathsf{A}a^{-1}\mathsf{B} \mid b\mathsf{B}b^{-1}\mathsf{B} \mid a^{-1}\mathsf{C}a\mathsf{B} \mid \epsilon$

$\mathsf{C} \to b\mathsf{B}b^{-1}\mathsf{C} \mid a^{-1}\mathsf{C}a\mathsf{C} \mid b^{-1}\mathsf{D}b\mathsf{C} \mid \epsilon$

$\mathsf{D} \to a\mathsf{A}a^{-1}\mathsf{D} \mid a^{-1}\mathsf{C}a\mathsf{D} \mid b^{-1}\mathsf{D}b\mathsf{D} \mid \epsilon$

Eg: $\mathsf{F}_2 = \langle a, b \mid \ \rangle$

$g(z) = 4z^2 f(z) g(z) + 1$ and $f(z) = 3z^2 f(z) f(z) + 1$

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so $3z^2 f(z)^2 - f + 1 = 0$ so $f(z) = \dfrac{1 - \sqrt{1 - 12z^2}}{6z^2}$

and $g(z) = \dfrac{1}{1 - 4z^2 f(z)}$ so $g(z) = \dfrac{3}{1 + 2\sqrt{1 - 12z^2}}$

which has radius of convergence $\frac{1}{\sqrt{12}}$ so $\rho = \sqrt{12} \approx 3.4$

Thm

(Grigorchuk/Cohen)

G is amenable iff $\rho = |X|$.

So ...

counting return paths in different graphs gives information about (the sometimes very difficult problem of) amenability.

Statistical physicists have studied similar problems: self-avoiding polymers, walks, etc;

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Statistical physicists have studied similar problems: self-avoiding polymers, walks, etc;

and good techniques have been developed.

The idea of this work is to apply such techniques to sample trivial words from a group and use statistical information from such experiments to find bounds and estimates for cogrowth

with one particular group in mind: R. Thomspon's group F.

Previous work

Belk 2005: upper bound of $\frac{1}{2}$ for isoperimetric constant $\inf_{n \to \infty} \frac{|\partial \mathsf{F}_n|}{|\mathsf{F}_n|}$

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Elder-Rechnitzer-Wong 2011: lower bounds for cogrowth by looking at number of loops in finite subgraphs of Cayley graphs

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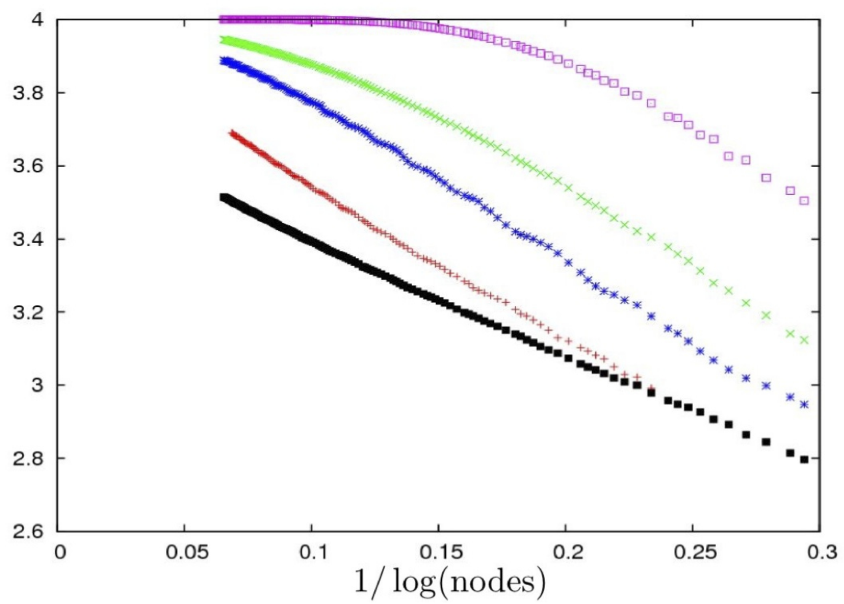
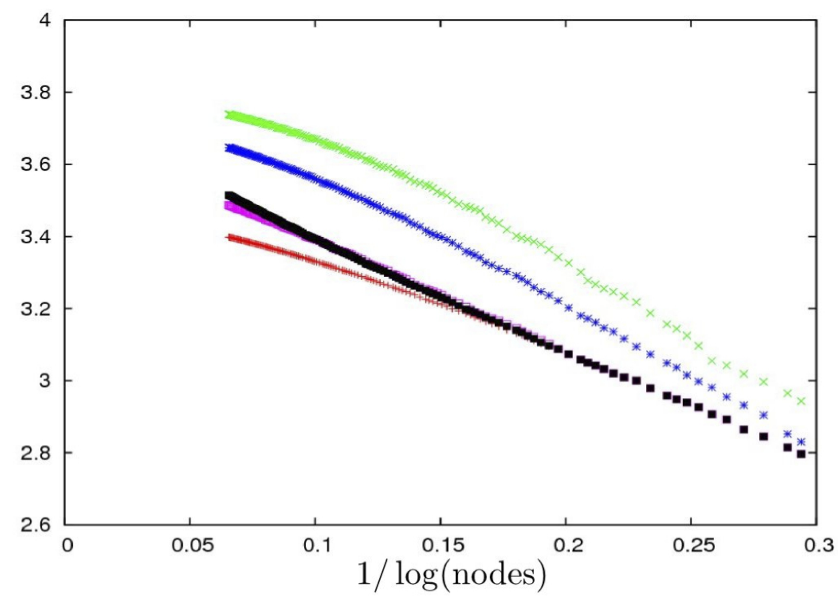


F (black) vs.
amenable groups
($\mathbb{Z} \times \mathbb{Z}$ is pink)



F (black) vs.
non-amenable groups
(F_2 is red)

Alternative defn of cogrowth

One can also count the number of **freely reduced** words equal to e, say $r(n)$

and put

$\overline{\rho} = \limsup_{n\to\infty} r(n)^{1/n}$ the **(reduced) cogrowth rate** for (G,X)

and $\sum_{n=0}^{\infty} r(n) z^n$ the **(reduced) cogrowth series** for (G,X)

Alternative defn of cogrowth

Thm (Grigorchuk/Cohen)

G is amenable iff $\overline{\rho} = |X| - 1$.

(Note the total number of reduced words on X=X^{-1} is $|X| - 1$).

Sampling trivial words

Let G be a group with finite presentation $\langle\, X = X^{-1} \mid R \,\rangle$

Let R_c be the set of all cyclic permutations of relators in R, and their inverses.

trivial word = freely reduced word equal to the identity

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Consider the following two moves on trivial words:

- conjugate by a generator then freely reduce
- insert at some position a word from R_c then freely reduce

Sampling trivial words

conjugation is a uniquely reversible operation

but insertion is not:

- $w = a^n r^{-1} a^{-n}$, inserting r gives e

but neither conj or insertion will recover w in one move

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solution: reject a move if it results in cancelation more than the length of r.

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- in $\mathbb{Z}^2$, $w = uba^{-1}b^{-1}\ aba^{-1}v$, inserting $bab^{-1}a^{-1}$ gives $uba^{-1}v$

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or: insert $b^{-1}aba^{-1}$ before v

solution: reject if inserting r leads to cancelations on the right of r

Left-insertions

Define a **left-insertion** on a trivial word w as follows:

for $r \in \mathsf{R}_c$ and $m \in \{0, 1, \ldots, |w|\}$

write $w = uv$ with $|u| = m$

set $w' = urv$

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set $w' = urv$

if rv freely cancels, reject

else freely reduce w', and reject if the result has length $< |w| - |r|$

Left-insertions

Lem

Left-insertions are uniquely reversible

Lem

Every trivial word can be obtained starting from $w_0 =$ some relator from R_c and applying some sequence of conjugations and left-insertions.

Markov Chain Monte Carlo sampling

Now the theory of MCMC applies:

1. define a probability distribution P over R_c (uniform if R is finite)

2. fix $p_c \in [0, 1]$ and $p = p(u, v, \alpha, \beta) \in [0, 1]$ where p depends on the trivial words u, v and constants $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^+$

3. start with a **state** (trivial word) w_0 from R_c chosen with probability $\mathsf{P}(w_0)$

Markov Chain Monte Carlo sampling

4. if w_n is the current state, with probability p_c choose a conjugation move, else choose a left-insertion

• if conjugation, choose one of the |X| possible conjugations uniformly at random, put $u = cw_nc^{-1}$ and freely reduce to obtain w'

put $w_{n+1} = \begin{cases} w' & \text{with probability } p(w, w', \alpha, \beta) \\ w_n & \text{otherwise} \end{cases}$

Markov Chain Monte Carlo sampling

• if left-insersion, choose $r \in \mathsf{R}_c$ with probability $\mathsf{P}(r)$, a location $m \in [0, \ldots, |w_n|]$ uniformly, and let w' be the outcome of the insertion

$$\text{put } w_{n+1} = \begin{cases} w_n & \text{if insertion invalid} \\ w' & \text{with probability } p(w, w', \alpha, \beta) \\ w_n & \text{otherwise} \end{cases}$$

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Note that we only allow valid moves with a certain probability, which depends on a constant β.

(This option to reject a move with some probability is called the **Metropolis** MCMC method.)

Markov Chain Monte Carlo sampling

Varying the value of β changes the expected value of the mean length of words sampled.

Analysis of the probabilities shows there is a critical value, β_c, for β below which the expected mean length is finite and above it is infinite.

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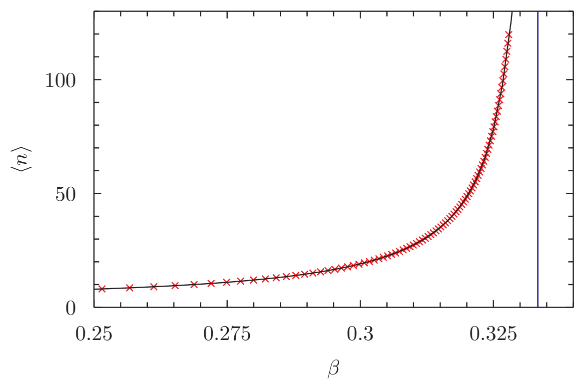
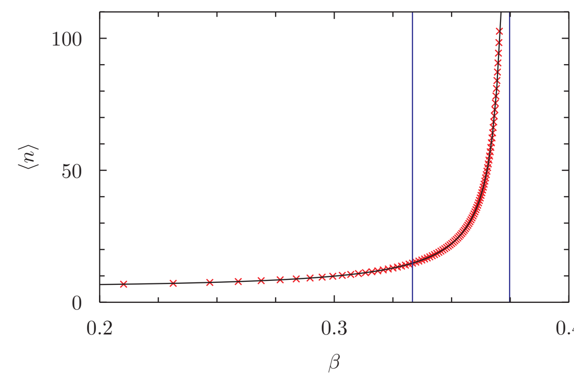
The cogrowth rate corresponds to the **reciprocal** of this value.

Details (and **results**) in our paper.

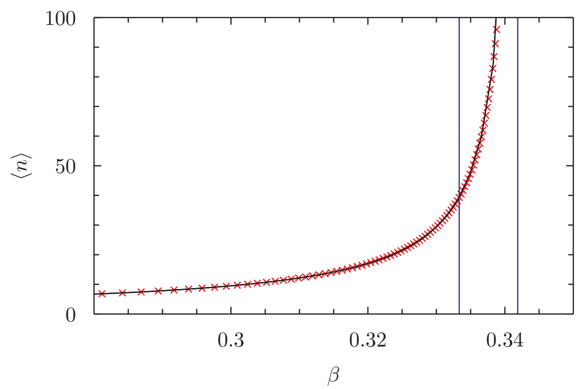
Results



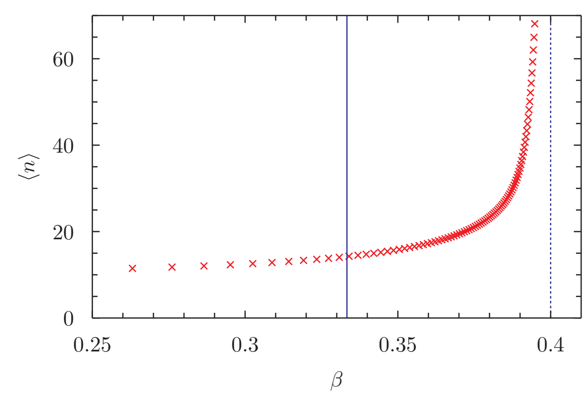
$\mathbb{Z}^2$



$\mathsf{BS}(2,2)$



$\mathbb{Z}_2 * \mathbb{Z}_3$



F

Mean length of freely reduced trivial words at different values of β. The solid blue lines indicate the reciprocal of the cogrowth of amenable groups with $|X| = 4$, $\beta_c = \frac{1}{3}$. The dashed blue lines indicate the approximate location of the vertical asymptote.

Part 2: lower bounds for growth and metric estimates

$\mathsf{E} : \mathsf{G} \to \mathbb{R}$ is a *metric estimate* for a group G if there are constants C_1, C_2 so that

$$\mathsf{C}_1\mathsf{E}(x) \leq |x| \leq \mathsf{C}_2\mathsf{E}(x)$$

In addition, a *good* metric estimate is one that is easy to compute.

Estimates have been used to study subgroup distortion, etc.

metric estimate for $\mathsf{BS}(m,n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle$

each element can be written in the form $\mathsf{P}a^{\mathsf{N}}$ where $\mathsf{P} \in \{t, at, \ldots, a^{n-1}t, t^{-1}, at^{-1}, \ldots, a^{m-1}t^{-1}\}^*$

by applying the moves $a^m t^{-1} \to t^{-1} a^n$ and $a^n t \to t a^m$

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Eg: $\mathsf{BS}(2,2)$

$a^{-1}ta^{-1}ta^{-1}ta^{-1}ta^{-1}t$

metric estimate for BS$(m,n) = \langle a,t \mid ta^mt^{-1} = a^n \rangle$

each element can be written in the form Pa^{N} where P $\in \{t, at, \ldots, a^{n-1}t, t^{-1}, at^{-1}, \ldots, a^{m-1}t^{-1}\}^*$

by applying the moves $a^mt^{-1} \to t^{-1}a^n$ and $a^nt \to ta^m$

Eg: BS$(2,2)$

$$a^{-1}ta^{-1}ta^{-1}ta^{-1}ta^{-1}t = aa^{-2}ta^{-1}ta^{-1}ta^{-1}ta^{-1}t$$

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$= ata^{-1}ta^{-1}ta^{-1}ta^{-1}ta^{-2}$

$\ldots = atatatatata^{-10}$

metric estimate for $\mathrm{BS}(m,n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle$

Assume $m \leq n$.

We can write $a^{\mathrm{N}} = a^{r_0} t a^{r_1} t \ldots a^{r_s} t a^{\kappa} t^{-s}$

where $0 < \kappa < n, 0 \leq r_i < n$ (or $-n < \kappa < 0, -n < r_i \leq 0$ if N< 0)

and s is at most $\log_{n/m} |\mathrm{N}|$

check

a^{100} in BS(2, 3):

$aa^{99} = ata^{66}t^{-1} = atta^{44}t^{-2} = atta^2ta^{28}t^{-3} = atta^2tata^{18}t^{-4} = atta^2tatta^{12}t^{-5} = atta^2tattta^8t^{-6} = atta^2tattta^2ta^4t^{-7}$

$$\log_{3/2} 100 = \frac{\log_e 100}{\log_e 3/2} = 11.3577...$$

metric estimate for $\mathsf{BS}(m,n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle$

So $x =$PQ where P is as before and Q is of the form

$$a^{r_0} t a^{r_1} t \ldots a^{r_s} t a^{\kappa} t^{-s}$$

which has length some constant multiple of $\log_{n/m} |\mathsf{N}|$

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Propn (Elder, Burillo)

$\mathrm{E}(x) = |\mathrm{P}| + \log |\mathrm{N}|$

is a metric estimate for BS(m,n).

metric estimate for $\mathrm{BS}(m,n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle$

Proof: The upper bound is clear – we can write x =PQ and PQ has this length (up to a constant)

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For the lower bound, if $w = \mathsf{P}a^{\mathsf{N}}$ then multiplying on the right by $a^{\pm 1}$ increases $\log \mathsf{N}$ by at most 1,

and multiplying on the right by $t^{\pm 1}$ increases $|\mathsf{P}|$ by at most $\max\{|m|, |n|\} = c$ and $\log \mathsf{N}$ by at most 1.

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and multiplying on the right by $t^{\pm 1}$ increases $|\mathsf{P}|$ by at most $\max\{|m|, |n|\} = c$ and $\log \mathsf{N}$ by at most 1.

So if x has geodesic length $|x|$, each time we multiply by a letter (starting at e), $|\mathsf{P}|$ increases by at most c and $\log \mathsf{N}$ by at most 1, so after $|x|$ letters we have $|\mathsf{P}| + |\log \mathsf{N}| \leq c|x| + |x|$

lower bounds for growth

We are currently using these estimates to find good lower bounds for the growth function

i.e. if $\mathsf{C}_2\mathsf{E}(x) \leq r$ then $x \in \mathsf{B}(r)$

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