Geodesic growth in right-angled Artin and even Coxeter groups

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University of Neuchâtel, Switzerland

Webinar, Nov. 8, 2012

Motivation

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Answer: (A–C 2012) Yes.

Preprint: `http://arxiv.org/abs/1203.2752`

Broader theme: geodesic rigidity of groups

Let G_1 and G_2 be in same family of groups (Coxeter, Artin etc.).

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- What kind of combinatorial properties do the graphs that define G_1 and G_2 have in common?
- How does standard (spherical) growth compare to geodesic growth from the point of view of rigidity?

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- even Coxeter groups.

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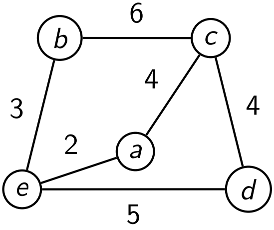

$$\left\langle a, b, c, d, e \;\middle|\; \begin{array}{c} a^2 = b^2 = c^2 = d^2 = e^2 = 1 \\ (bc)^6 = (cd)^4 = (de)^5 = 1 \\ (eb)^3 = (ea)^2 = (ac)^4 = 1 \end{array} \right\rangle$$

RACGs and RAAGs

- The system is even if m only takes even values.
- The system is right-angled if m only takes the value 2.

- The right-angled Coxeter group (RACG) G determined by Γ is

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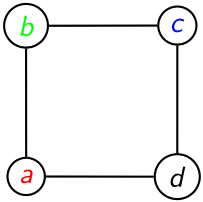
Example



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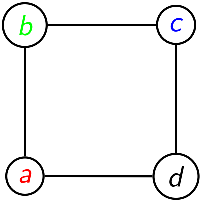

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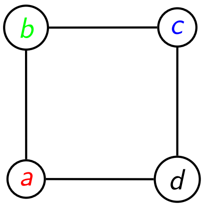
Elements	Geodesics

Example



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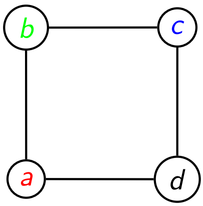
Elements

- 1
-
-
-

Geodesics

- 1
-
-
-

Example



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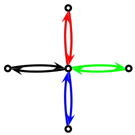

Elements

- 1
- 4
-
-

Geodesics

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- 4
-
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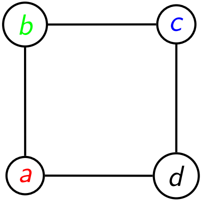
Example



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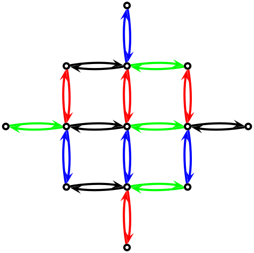


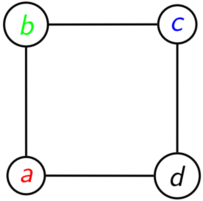
Elements

- 1
- 4
- 8
-

Geodesics

- 1
- 4
- 12
-

Example



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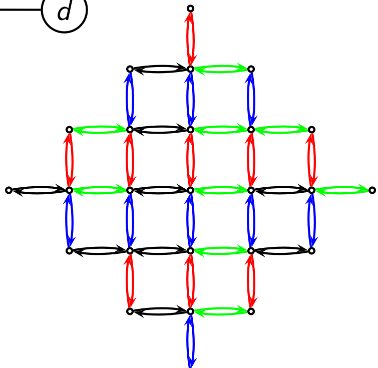

Elements

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Geodesics

- 1
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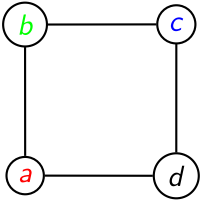
- 28

Example



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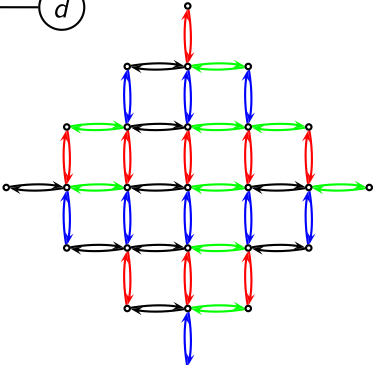

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$28 = |V|(|V| - 1)^2 - 2|E|$

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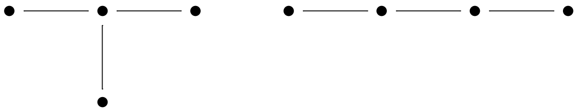
Theorem (Steinberg (1968))

$$\frac{1}{\mathcal{S}(G_\Gamma)(z)} = f_\Gamma\left(\frac{-z}{1+z}\right)$$

The spherical growth of RACGs and RAAGs

- Only depends on the f-polynomial of the simplicial graph.

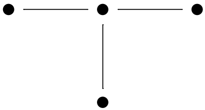
Ex: Two trees with the same number of vertices have the same spherical growth.

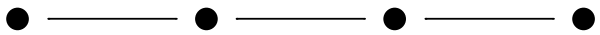


$$\boxed{f(x) = 1 + 4x + 3x^2}$$

The geodesic growth of RACGs and RAAGs

- There exist graphs with same f-polynomial but different geodesic growth.





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- Changing the generating sets will modify all statements above.

Main questions

Can two non-isomorphic RAAGs (or RACGs) have the same geodesic growth?

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Theorem (A – C)

Let Γ be a link-regular graph. Then the geodesic growth of the right-angled Coxeter (or Artin) group based on Γ is a function of the sizes of the links and the f-polynomial.

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Theorem (A – C)

Let (Γ, m) be an even Coxeter system with Γ triangle-free and star-regular. Then $\mathcal{G}(\Gamma)$ is a function of the star of a vertex and $|V|$.

Corollary. *Let G and G' be two right-angled Artin or Coxeter groups that are link-regular and have the same f-polynomial. Then G and G' have the same geodesic growth.*

The smallest example

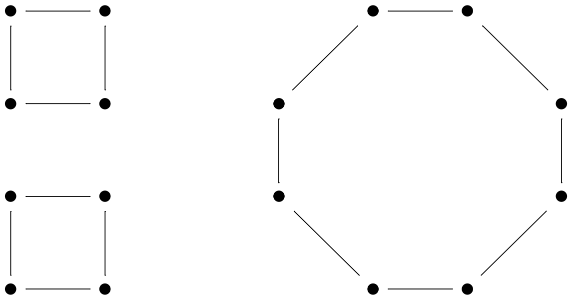

Figure: Two RACGs or RAAGs with the same geodesic growth

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For a RAAG: use a result of Droms and Sevatius that connects Cayley graphs of RACSs and RAAGs.

Automaton recognizing geodesics in a RACG

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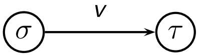

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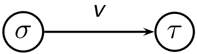

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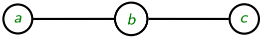
Example (

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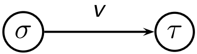

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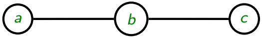
Example (

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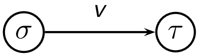

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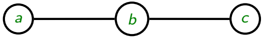

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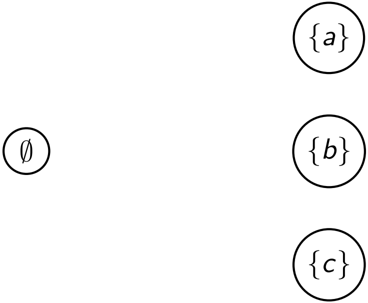

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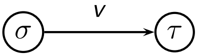

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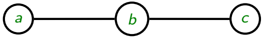
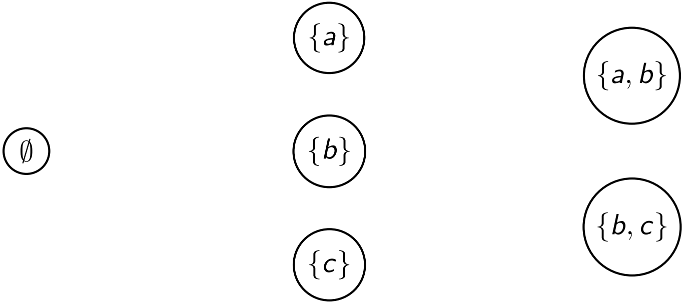
Example ()



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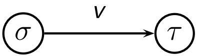

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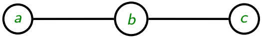

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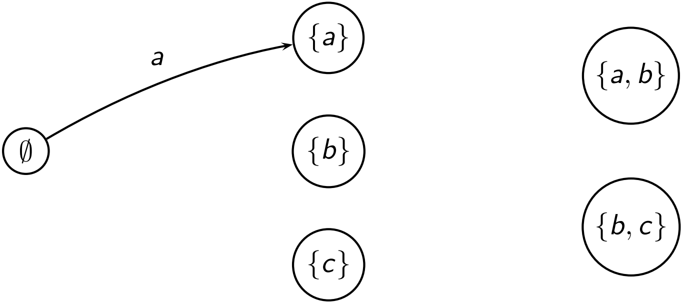

Automaton recognizing geodesics in a RACG

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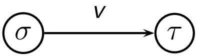

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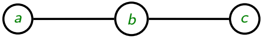
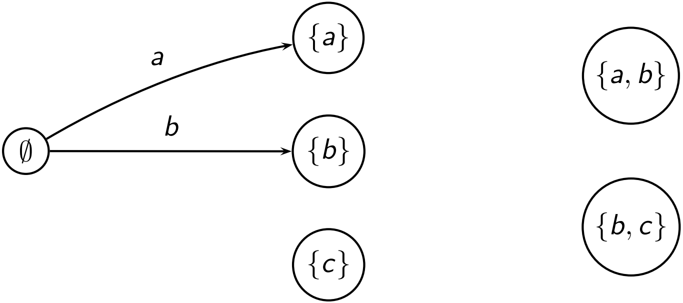
Example ()



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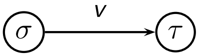

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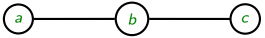
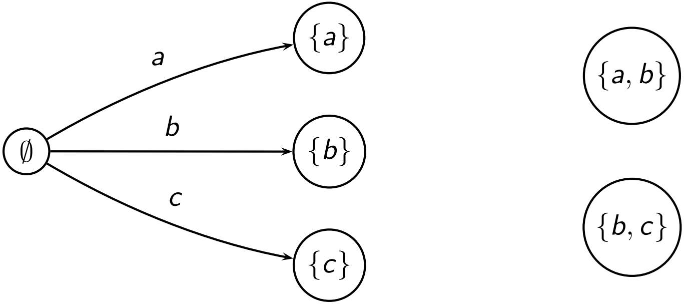
Example ()



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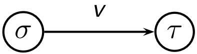

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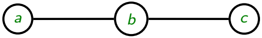
Example (

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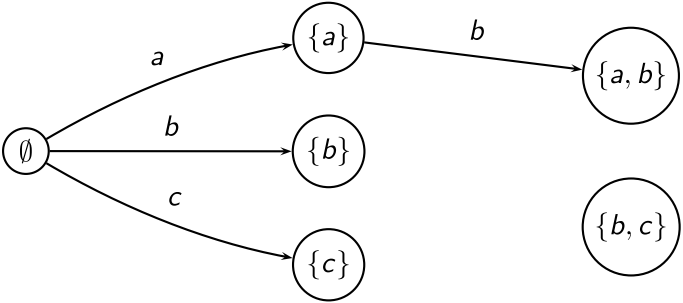

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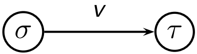

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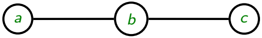
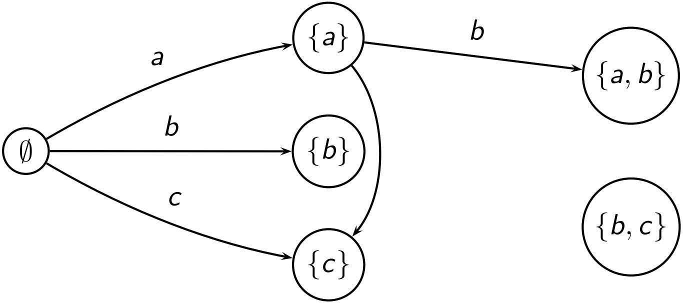
Example ()



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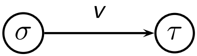

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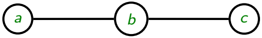

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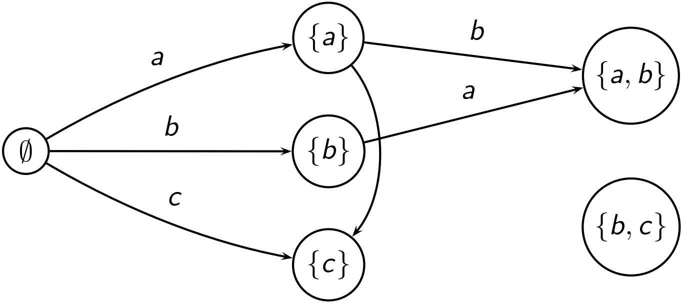

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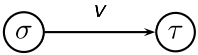

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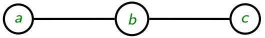
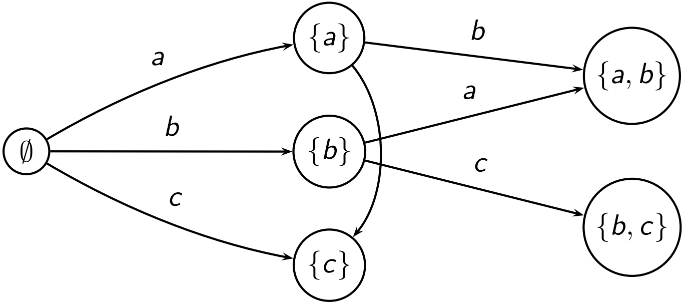
Example ()



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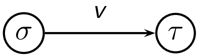

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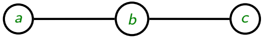
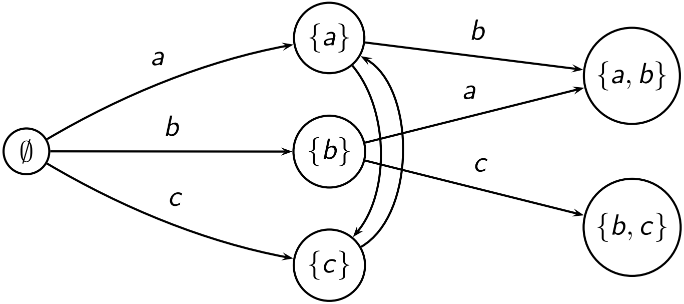
Example ()



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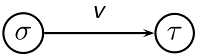

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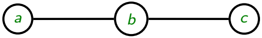
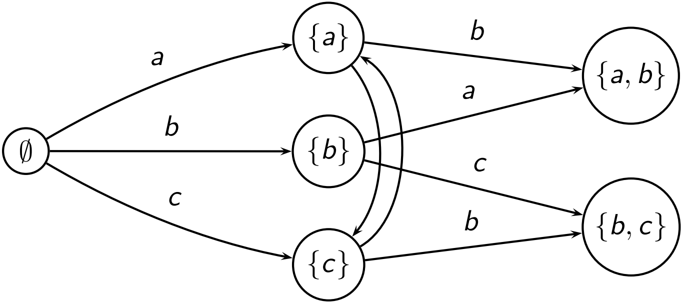
Example ()



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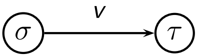

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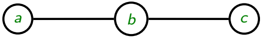
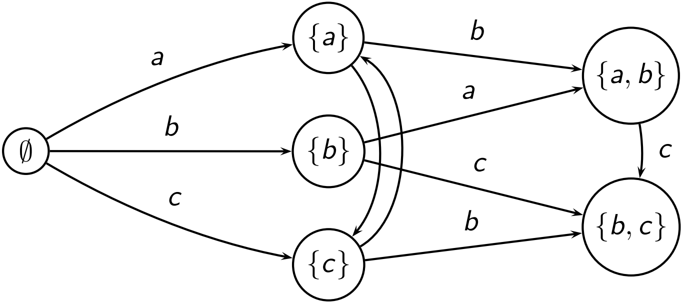
Example ()



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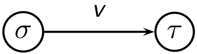

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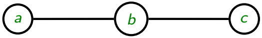
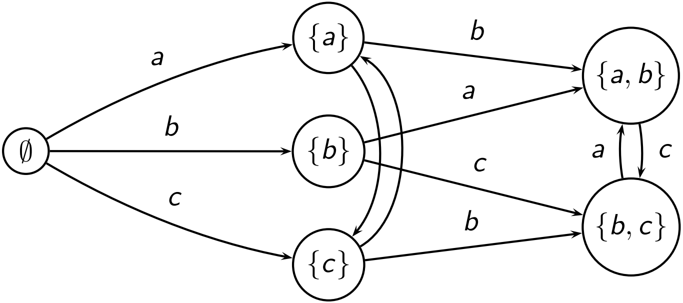
Example ()



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Regular graph implies 'nice' automaton

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Lemma

If Γ is link-regular, then $\deg_j(\sigma)$ depends only on $|\sigma|$, j and $|\operatorname{Lk}(\tau)|$, $\tau \in \mathcal{A}$.

- So we can write $\beta_{\mathbf{i,j}} = \sharp$ transitions from any fixed i-state to all j-states.

Corollary

If Γ is link-regular, $\beta_{\mathbf{i,j}}$ only depends on the f-polynomial and $|\operatorname{Lk}(\tau)|$, $\tau \in \mathcal{A}$.

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Example: For the group

$$G = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, [a, b] = [b, c] = 1\rangle,$$

$$A = \{a, b, c, ab, bc\}.$$

RACGs with same automatic growth

Theorem [R. Glover and R. Scott, *Involve*, 2009].

Let G and G' be two right-angled Coxeter groups with link-regular nerves and same f-polynomial. Then G and G' have the same automatic growth.

Questions

If two RACGs have the same geodesic growth, does it imply that they have the same automatic growth, and vice versa?

The Theorem for RAAGs

Droms and Servatius: the Cayley graph of the RAAG based on a graph Γ is isomorphic as undirected graph to the Cayley graph of the RACG based on Γ^2, the double of Γ :

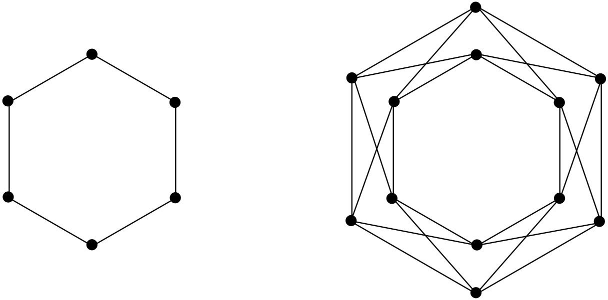

Figure: The hexagon and the double of the hexagon.

Even Coxeter groups

Theorem 2. [A – C]

Let (W, S) be an even Coxeter system with graph Γ, where Γ is triangle-free and star-regular. The geodesic growth of W depends only on $|S|$ and the isomorphism class of the star of the vertices.

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In particular, if (W_1, S_1) and (W_2, S_2) are triangle-free, star-regular, even Coxeter systems with $|S_1| = |S_2|$ and $St(v) \cong St(u)$, $\forall v \in V\Gamma_1$, $u \in V\Gamma_2$, then W_1 and W_2 have the same geodesic growth.

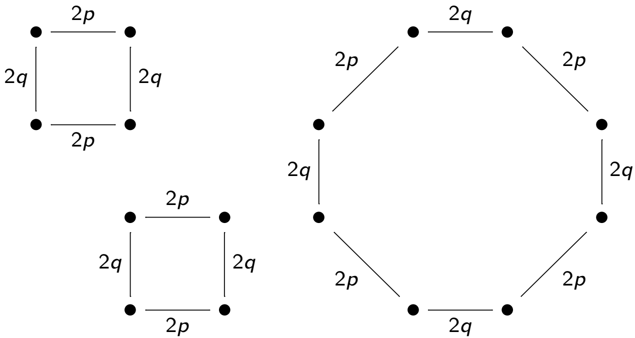

Figure: Two even Coxeter groups with the same geodesic growth

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- Understand centralizers of generators in even Coxeter groups.
- Construct and analyze the automata recognizing geodesics in even Coxeter groups.
- For 'nice' regular graphs Γ, these automata also have 'nice' graph-theoretic properties, and one can simplify the counting as in the previous theorem.

Centralizers in even Coxeter groups

The main obstruction to being geodesic is containing a subword in

$$U = \{vwv | w \text{ is a geodesic word in } \mathbf{C}_G(v),\ v \in V\}$$

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$$a_{t,s} := tst \ldots st$$

of length $m(\{t, s\}) - 1$*. Then*

- *(Brink)* $\mathbf{C}_G(s) = \langle\, s \mid s^2 \,\rangle \times \langle\, a_{t,s}, \{t, s\} \in E \mid a_{t,s}^2 \,\rangle$.

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The Automaton

For a Coxeter system (G, S), let $w(s, t; m_{s,t})$ be the left-hand side of the relation involving generators s, t. Define

$$\boxed{Z(s, t, m) \coloneqq \{g |\ g \text{ is a right-divisor of } w, w \equiv w(s, t; m)\}}$$

- The states of the automaton are in bijection with the sets $\sigma = Z(s, t, m)$, where $s, t \in S$ and $m \leq m_{s,t}$.
- The transition is given by $g \xrightarrow{v} h$, $g \in \sigma$, if $|gv| = |g| + 1$ and h is a maximal alternating right-divisor of gv.

Open questions

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