Free affine actions on linear Λ-trees

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I will only look at actions on lines today.

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Therefore a group admits a free isometric action (without inversions) on a line if and only if it is torsion-free abelian.

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An affine automorphism ϕ of a real metric space X is a surjective function $X \to X$ for which there exists $\alpha = \alpha_\phi \in \mathbb{R}$ such that

$$d(\phi x, \phi y) = \alpha_\phi d(x, y).$$

Such an α_ϕ must be positive (assuming $|X| > 1$).

Theorem (I. Liousse (2001))

There are groups that admit free affine actions on $\mathbb{R}$-trees that don't admit free isometric actions on any $\mathbb{R}$-tree. An example is

$$\Gamma_0 = \langle x_1, x_2, x_3, y_1, y_2, y_3 \mid [x_1, y_1] = [x_2, y_2] = [x_3, y_3] \rangle.$$

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In earlier work we have shown

Theorem (A. Martino, SOR (2004))

Liousse's groups do admit a free isometric action on a $\mathbb{Z}^n$-tree for some n.

Recall that $\phi : X \twoheadrightarrow X$ is an affine automorphism if there exists $\alpha = \alpha_\phi \in \mathbb{R}$ such that

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So let

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be a homomorphism (Aut^+ denotes the group of order-preserving automorphisms). An α-affine action of G on a Λ-tree X is an action satisfying

$$d(gx, gy) = \alpha_g d(x, y) \ \ \forall x, y \in X$$

Some features of affine actions on general Λ-trees.

1. The based length function (Lyndon length function) $L_x : g \mapsto d(x, gx)$ can be defined and in fact determines an affine action much as in the isometric case. (The hyperbolic length function does *not* generalise usefully however.)

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2. The class ATF of groups that admit a free affine action on a Λ-tree for some Λ is closed under free products and ultraproducts.
3. As in the isometric case, a group G is
   - locally in ATF or
   - fully residually in ATF

   if and only if G is in ATF.

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Define an action of $\Gamma = \langle a, t \rangle$ on $\mathbb{Z} \times \mathbb{R}$ via

$$\begin{aligned} a \cdot (m, x) &= (m, & x + 1) \\ t \cdot (m, x) &= (m + 1, & rx). \end{aligned}$$

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This action is also rigid in the sense that $g[x, y] \subseteq [x, y]$ implies $g[x, y] = [x, y]$ (and hence $g = 1$ since the action is free).

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But the natural action of $\mathrm{UT}(n+1, \mathbb{Z})$ on $\mathbb{Z}^n$ is not free.

Call a matrix $A \in \mathrm{UT}(m+1, \mathbb{Z})$ (or even $T(m+1, \mathbb{R})$) admissible if $A = I$ or if the lowest non-zero entry of $A - I$ lies in the last column and is strictly lower than any other non-zero entry.

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So $A \neq I$ is admissible if and only if A fixes no point and is rigid.

Question: Which groups admit a representation as admissible matrices in $\mathrm{UT}(m+1, \mathbb{Z})$ for some m?

Example: Consider $x : (n_1, n_2, n_3) \mapsto (n_1, n_2 + 1, n_3)$ and $y : (n_1, n_2, n_3) \mapsto (n_1 + 1, n_2, n_3 + n_2)$.

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We can represent x and y by matrices as follows.

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{x} \begin{pmatrix} n_3 \\ n_2 \\ n_1 \\ 1 \end{pmatrix} = \begin{pmatrix} n_3 \\ n_2 + 1 \\ n_1 \\ 1 \end{pmatrix}$$

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This gives a representation of $\langle x, y \rangle$ as admissible matrices in $\mathrm{UT}(4, \mathbb{Z})$, and thus a free rigid affine action on $\mathbb{Z}^3$.

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Question: Do all unitriangular groups $\mathrm{UT}(n, \mathbb{Z})$ admit a faithful representation as admissible matrices?

Hint (K. Dekimpe): Look at affine structures on $\mathrm{UT}(n, \mathbb{Z})$, left symmetric algebras.

(See K. Dekimpe, W. Malfait
'Affine structures on a class of virtually nilpotent groups',
Top. Appl. 1996 for more details.)

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  Extend to a binary operation on $\mathfrak{g}$ using bilinearity. This gives a left symmetric structure on $\mathfrak{g}$. That is, $\cdot$ is a bilinear operator satisfying

  1. $[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$;
  2. $[x, y] = x \cdot y - y \cdot x$.

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For example, if $n = 4$ then $t : \begin{pmatrix} 0 & s & v & w \\ 0 & 0 & r & u \\ 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} w \\ \hline v \\ u \\ \hline s \\ r \\ q \end{pmatrix}$.

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Then $\lambda(x)$ is an $m \times m$ upper triangular matrix.

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  Then $d\bar{\gamma}$ is a complete affine structure meaning that

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  This defines $d\bar{\gamma} : \mathfrak{ut}(n, \mathbb{Q}) \to \mathfrak{ut}(m+1, \mathbb{Q})$.
- Let $\bar{\gamma} : g \mapsto \exp \cdot d\bar{\gamma} \cdot \log(g)$

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Proposition

$\bar{\gamma} : \mathrm{UT}(n, \mathbb{Q}) \to \mathrm{UT}(m+1, \mathbb{Q})$ *is an injective group homomorphism with admissible image.*

Example: If $n = 3$ and $x_i = \begin{pmatrix} 0 & s_i & v_i \\ 0 & 0 & r_i \\ 0 & 0 & 0 \end{pmatrix}$ $(i = 1, 2)$, then

$$x_1 \cdot x_2 = \begin{pmatrix} 0 & 0 & \frac{r_2 s_1}{2} - \frac{r_1 s_2}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$t(x_2) = \begin{pmatrix} v_2 \\ s_2 \\ r_2 \end{pmatrix} \quad t(x_1 \cdot x_2) = \begin{pmatrix} \frac{r_2 s_1}{2} - \frac{r_1 s_2}{2} \\ 0 \\ 0 \end{pmatrix}$$

This gives $\lambda(x_1) = \begin{pmatrix} 0 & -\frac{r_1}{2} & \frac{s_1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

and hence $d\bar{\gamma}(x_1) = \left(\begin{array}{ccc|c} 0 & -\frac{r_1}{2} & \frac{s_1}{2} & v_1 \\ 0 & 0 & 0 & s_1 \\ 0 & 0 & 0 & r_1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$.

It follows that if $g = \begin{pmatrix} 1 & s & v \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}$ then

$$\bar{\gamma}(g) = \exp \cdot d\bar{\gamma} \cdot \log(g) = \begin{pmatrix} 1 & -r/2 & s/2 & v - rs/2 \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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3. Every finitely generated torsion-free nilpotent group embeds in $\mathrm{UT}(n, \mathbb{Z})$ for some n. (P. Hall)

Theorem

The groups that admit free affine actions on $\mathbb{Z}^n$ for some n are precisely finitely generated torsion-free nilpotent groups.

Corollary

1. *Every locally residually torsion-free nilpotent group admits a free rigid affine action on a line.*
2. *Every free polynilpotent group (of given class row) admits a free rigid affine action on a line.*

Recall (once more) that $BS(1, r)$ admits a free rigid action on $\mathbb{Z} \times \mathbb{R}$, via

$$\begin{array}{lll} a \cdot (m, x) & = (m, & x + 1) \\ t \cdot (m, x) & = (m + 1, & rx). \end{array}.$$

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$$a \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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So what other (non-nilpotent) groups of upper triangular matrices admit free affine actions on $\mathbb{R}^n$ for some n?

Let $B = T^*(n, \mathbb{R})$ denote the group of all upper triangular matrices with real entries and positive diagonal entries.

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Theorem

The group $T^*(n, \mathbb{R})$ admits an embedding in $T^*(m+n+1, \mathbb{R})$ with admissible image. Thus $T^*(n, \mathbb{R})$ admits a free rigid affine action on $\mathbb{R}^{m+n}$ (considered as an $\mathbb{R}^{m+n}$-tree).

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The proof loosely follows an argument of John Milnor (see the proof of Theorem 1.2 in 'On Fundamental Groups of Complete Affinely Flat Manifolds' (Adv. Math. 1977)).

We already have an admissible embedding $\varphi = \bar{\gamma} : U \to \mathrm{UT}(m+1, \mathbb{R})$. Write

$$\varphi(u) = \begin{pmatrix} \varphi_0(u) & b(u) \\ 0 & 1 \end{pmatrix}$$

where $\varphi_0(u) \in \mathrm{UT}(m, \mathbb{R})$ and $b(u) \in \mathbb{R}^m$.

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where $\varphi_0(u) \in \mathrm{UT}(m, \mathbb{R})$ and $b(u) \in \mathbb{R}^m$. For $d = diag(d_1, \ldots, d_n)$, let $d^* = diag(\frac{d_1}{d_n};\ \frac{d_1}{d_{n-1}}, \frac{d_2}{d_n};\ \ldots; \frac{d_1}{d_2}, \frac{d_2}{d_3}, \ldots \frac{d_{n-1}}{d_n})$, an $m \times m$ diagonal matrix.

Let $\log(d)$ denote the *column vector* $(\log d_1, \ldots, \log d_n)^T$.

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Now define $\bar{\varphi}(u) = \begin{pmatrix} \varphi_0(u) & 0 & b(u) \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix}$

and

$$\bar{\varphi}(d) = \begin{pmatrix} d^* & 0 & 0 \\ 0 & I_n & \log(d) \\ 0 & 0 & 1 \end{pmatrix}$$

Then

Proposition

1. $\bar{\varphi}(dud^{-1}) = \bar{\varphi}(d)\bar{\varphi}(u)\bar{\varphi}(d^{-1})$.
2. $\bar{\varphi} : T^*(n,\mathbb{R}) \to T^*(m+n+1,\mathbb{R})$ *is an injective homomorphism with admissible image.*

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Proposition

1. $\bar{\varphi}(dud^{-1}) = \bar{\varphi}(d)\bar{\varphi}(u)\bar{\varphi}(d^{-1})$.
2. $\bar{\varphi} : T^*(n, \mathbb{R}) \to T^*(m + n + 1, \mathbb{R})$ *is an injective homomorphism with admissible image.*

Consequently,

Theorem

$T^*(n, \mathbb{R})$ *admits a free rigid affine action on* $\mathbb{R}^{m+n}$.

Example: $n = 3$

A typical element of $T^*(3, \mathbb{R})$ is expressible in the form ud where

$$u = \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \text{ and } d = \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix}.$$

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$$\text{Now } \varphi(u) = \left(\begin{array}{ccc|c} 1 & -x/2 & y/2 & z - xy/2 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \text{ so that}$$

$$\varphi_0(u) = \begin{pmatrix} 1 & -x/2 & y/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } b(u) = \begin{pmatrix} z - xy/2 \\ y \\ x \end{pmatrix}.$$

Also $d^* = \begin{pmatrix} r/t & 0 & 0 \\ 0 & r/s & 0 \\ 0 & 0 & s/t \end{pmatrix}$ so that $\bar{\varphi}(ud) = \bar{\varphi}(u)\bar{\varphi}(d)$

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Go raibh maith agaibh!