Invariant Random Subgroups

Lewis Bowen

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Set-up

$G$: a locally compact group.

$\text{Sub}(G)$: the space of closed subgroups.

$G$ acts on $\text{Sub}(G)$ by conjugation.

$g \cdot H := gHg^{-1}$.

$M(G)$: space of $G$-invariant Borel probability measures on $\text{Sub}(G)$.

An invariant random subgroup (IRS) is a random subgroup $H < G$ with law in $M(G)$.
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3. Let \( \Gamma \triangleleft G \) be a lattice. \( \exists \) a \( G \)-invariant prob. meas. \( \lambda \) on \( G/\Gamma \).

Define \( \Phi : G/\Gamma \to \text{Sub}(G) \) by \( \Phi(g\Gamma) := g\Gamma g^{-1} \).
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  Then $\Phi_* \lambda := \mu_\Gamma \in M(G)$.
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  $\text{Stab}$ is $G$-equivariant $\Rightarrow \text{Stab}_* \mu \in M(G)$. 

(Abott-Glasner-Virag) $\Rightarrow$ every measure in $M(G)$ arises this way.
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  (Abert-Glasner-Virag) \( \Rightarrow \) every measure in \( M(G) \) arises this way.
$M(G)$ is a simplex

**Definition**

A convex closed metrizable subset $K$ of a locally convex linear space is a **simplex** if each point in $K$ is the barycenter of a unique probability measure supported on the subset $\partial_e K$ of extreme points of $K$. 
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If $\mu_1, \mu_2 \in M(G)$ and $t \in [0, 1]$ then $t\mu_1 + (1 - t)\mu_2 \in M(G)$.
Research directions in IRS’s

**Problem**: classify the ergodic IRS’s of a given group or describe $M(G)$. 

Remark 1. $M(G)$ is compact in the weak* topology. So it can be viewed as a compactification of the set of lattice subgroups.

Remark 2. If $K$ is an IRS then $K \setminus G$ can be thought of as something like a group. Although it need not be homogeneous, it possesses “statistical homogeneity”.

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Higher rank simple Lie groups

Theorem (Stuck-Zimmer, 1994)

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Let $X = G/K$. An $X$-manifold $M$ is a manifold locally modeled on $X$ (i.e., $M = X/\Gamma$ for some lattice $\Gamma < G$).
Higher rank simple Lie groups

**Theorem**

(Abert-Bergeron-Biringer-Gelander-Nikolov-Raimbault-Samet)

*If $G$ is as above, and $M_i$ is a sequence of $X$-manifolds such that*

\[
\lim_{i \to \infty} \text{vol}(M_i) = +\infty, \quad \liminf_{i \to \infty} \text{injrad}(M_i) > 0
\]

*⇒ $\forall k$, \ \ \ \lim_{i \to \infty} \frac{b_k(M_i)}{\text{vol}(M_i)} = \beta_k(X)$.*

**Sketch.**

Let $M_i = X/\Gamma_i$. By Stuck-Zimmer, $\mu_{\Gamma_i}$ converges in $M(G)$ to $\delta_e$. Show that $L^2$-betti numbers vary continuously on $M(G)$ using a generalized version of Lück approximation.
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$Z_n = X_1 \cdots X_n.$

$\{ Z_n \}$ is the simple random walk on $G$ with $\mu$-increments.
Entropy

Let $\mu^n$ be the law of $Z_n$, 

\[ H(\mu^n) := -\sum_{g \in G} \mu^n(\{g\}) \log \mu^n(\{g\}) \]

\[ h_{\mu^n}(G) := \lim_{n \to \infty} \frac{1}{n} H(\mu^n) \]

$0 \leq h_{\mu^n}(G) \leq h_{\mu^n}(F_2)$. 

Problem: What are all possible values of $h_{\mu^n}(G)$ as $G$ varies over all 2-generator groups?
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**Theorem**

\[ \{ h_\mu(G) : G \text{ a 2-generator group} \} \]

is dense in \([0, h_\mu(F_2)]\).
Random walks on random coset spaces

For $K < \mathbb{F}_2$, consider the random walk $\{KZ_n\}_{n=1}^{\infty}$ on $K \backslash \mathbb{F}_2$. 
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Let $\mu^n_K(\{Kg\}) = \text{Prob} (KZ_n = Kg)$,

$$h_\mu(\lambda) := \lim_{n \to \infty} \frac{1}{n} \int H(\mu^n_K) \ d\lambda(K).$$
Theorem

There exists a path-connected subspace $\mathcal{N} \subset M_{e}(F_2)$ on which the map $\lambda \in \mathcal{N} \mapsto h_\mu(\lambda)$ is continuous and surjects onto $[0, h_\mu(F_2)]$. 

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The finitely-supported measures in $\mathcal{N}$ are dense and these correspond to normal subgroups of $F_2$. Therefore, 

$$\{h_\mu(G) : G \text{ a 2-generator group}\}$$

is dense in $[0, h_\mu(F_2)]$. 
Let $K_n < F_2$ be the group generated by all elements of the form $ghg^{-1}$ where $g \in \langle a^n, b^n \rangle$ and either $h = a^k b^r a^{-k}$ for some $1 \leq |k| \leq n - 1$ and $r \in \mathbb{Z}$ or $h = b^k a^r b^{-k}$ for some $1 \leq |k| \leq n - 1$ and $r \in \mathbb{Z}$. 
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We can approximate \( \lambda_{n,p} \) be choosing a periodic collection of loops of \( K_n \setminus F_2 \) and then taking the universal cover of the 2-complex, which gives a Schreier coset graph for a group with only finitely many conjugates. Its normal core has entropy approximating \( \lambda_{n,p} \).
Classification Results

Theorem (Stuck-Zimmer, 1994)

\[ \text{If } G \text{ is a simple Lie group of real rank } \geq 2 \text{ and } K < G \text{ is an ergodic IRS then either } K \text{ is a lattice a.s. or } K = \{e\}. \]
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Theorem (Bader-Shalom, 2006)

If \( G_1, G_2 \) are just non-compact infinite property (T) groups then every ergodic IRS \( K < G_1 \times G_2 \) either splits as a product \( K = H_1 \times H_2 \) or \( K \) is a lattice subgroup a.s.
What sort of simplex is $M(G)$?

A simplex $\Sigma$ is

- **Poulsen** if $\partial_e \Sigma$ is dense in $\Sigma$;
- **Bauer** if $\partial_e \Sigma$ is closed in $\Sigma$.

**Theorem (Lindenstrauss-Olsen-Sternfeld, 1978)**

There is a unique Poulsen simplex $\Sigma$ up to affine isomorphism. Moreover, $\partial_e \Sigma \sim = L^2$.

There are uncountably many nonisomorphic Bauer simplices.
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Let $M_{ie}(\mathbb{F}_r) := M_e(\mathbb{F}_r) \setminus M_{fi}(\mathbb{F}_r)$ and $M_i(\mathbb{F}_r) = \text{Hull}(M_{ie}(\mathbb{F}_r))$. 
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**Theorem**

$M_i(\mathbb{F}_r)$ is a Poulsen simplex. So $M_{ie}(\mathbb{F}_r) \cong L^2$. 

Lewis Bowen (Texas A&M)
Given two Schreier coset graphs $K_1 \backslash F_r$, $K_2 \backslash F_r$, we can connect them together by replacing a vertex of each with 2 vertices and adding some edges.
Ergodic measures are dense

Let $\eta \in M_i(F_r)$.

For $\rho \in (0,1)$ we will construct $\eta_\rho \in M_{ie}(F_r)$ such that $\lim_{\rho \to 0} \eta_\rho = \eta$. 
Let $K < \mathbb{F}_r$ be random with law $\eta$. 
Building an ergodic approximation

Color each vertex of $K \backslash F_r$ red with prob. $p$ independently.
Building an ergodic approximation

At a red vertex, choose a random subgroup $L < F_r$ with law $\eta$ independent of $K$ and attach its Schreier coset graph by surgery to $K \backslash F_r$. 
At a red vertex, choose a random subgroup $J < \mathbb{F}_r$ with law $\eta$ independent of $K$ and other subgroups and attach its Schreier coset graph by surgery to $K \backslash \mathbb{F}_r$. 
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This is the Schreier coset graph of a random subgroup $K < F_2$. Let $\eta_p$ be the law of this subgroup. Show:

$\eta_p$ is ergodic and $\lim_{p \to 0} \eta_p = \eta$. 

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Further results and questions

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  $$\rho(K \setminus G) = \rho(G) \iff K \text{ is amenable a.s.}$$

- (B.) Any ergodic aperiodic probability-measure-preserving equivalence relation $(X, \mu, E)$ with $\text{cost}(E) < r$ is isomorphic to $(\text{Sub}(\mathbb{F}_r), \lambda, E_{\mathbb{F}_r})$ for some $\lambda \in M(\mathbb{F}_r)$. 
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- (Abert-Glasner-Weiss) If $K < G$ is an ergodic IRS then 
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  equivalence relation $(X, \mu, E)$ with cost$(E) < r$ is isomorphic to 
  $(\text{Sub}(\mathbb{F}_r), \lambda, E_{\mathbb{F}_r})$ for some $\lambda \in M(\mathbb{F}_r)$.

- (Bartholdi-Grigorchuk) There is a finitely generated group $G$ with 
  an ergodic IRS $K$ so that the Schreier coset graph $K \backslash G$ has 
  polynomial growth of irrational degree almost surely.