Invariant Random Subgroups

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Group Theory International Webinar
May 2012

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An *invariant random subgroup* (IRS) is a random subgroup $H < G$ with law in $M(G)$.

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  (Abert-Glasner-Virag) $\Rightarrow$ every measure in $M(G)$ arises this way.

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Definition

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If $\mu_1, \mu_2 \in M(G)$ and $t \in [0,1]$ then $t\mu_1 + (1-t)\mu_2 \in M(G)$.

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Remark 2. If K is an IRS then $K\backslash G$ can be thought of as something like a group. Although it need not be homogeneous, it possesses "statistical homogeneity".

Higher rank simple Lie groups

Theorem (Stuck-Zimmer, 1994)

If G is a simple Lie group of real rank ≥ 2 and $K < G$ is an ergodic IRS then either K is a lattice a.s. or $K = \{e\}$.

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Let $X = G/K$. An X-manifold M is a manifold locally modeled on X (i.e., $M = X/\Gamma$ for some lattice $\Gamma < G$).

Higher rank simple Lie groups

Theorem (Abert-Bergeron-Biringer-Gelander-Nikolov-Raimbault-Samet)

If G is as above, and M_i is a sequence of X-manifolds such that

$$\lim_{i\to\infty} vol(M_i) = +\infty, \quad \liminf_{i\to\infty} injrad(M_i) > 0$$

$$\Rightarrow \forall k, \quad \lim_{i\to\infty} \frac{b_k(M_i)}{vol(M_i)} = \beta_k(X).$$

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Sketch.

Let $M_i = X/\Gamma_i$. By Stuck-Zimmer, μ_{Γ_i} converges in $M(G)$ to δ_e. Show that L^2-betti numbers vary continuously on $M(G)$ using a generalized version of Lück approximation. □

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$\{Z_n\}$ is the simple random walk on G with μ-increments.

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Problem

What are all possible values of $h_\mu(G)$ as G varies over all 2-generator groups?

Random walk entropy

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Theorem

$$\{h_\mu(G) : \ G \textit{ a 2-generator group}\}$$

is dense in $[0, h_\mu(\mathbb{F}_2)]$.

Random walks on random coset spaces

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$$h_\mu(\lambda) := \lim_{n\to\infty} \frac{1}{n} \int H(\mu_K^n)\ d\lambda(K).$$

Random walk entropy

Theorem

There exists a path-connected subspace $\mathcal{N} \subset M_e(\mathbb{F}_2)$ *on which the map* $\lambda \in \mathcal{N} \mapsto h_\mu(\lambda)$ *is continuous and surjects onto* $[0, h_\mu(\mathbb{F}_2)]$.

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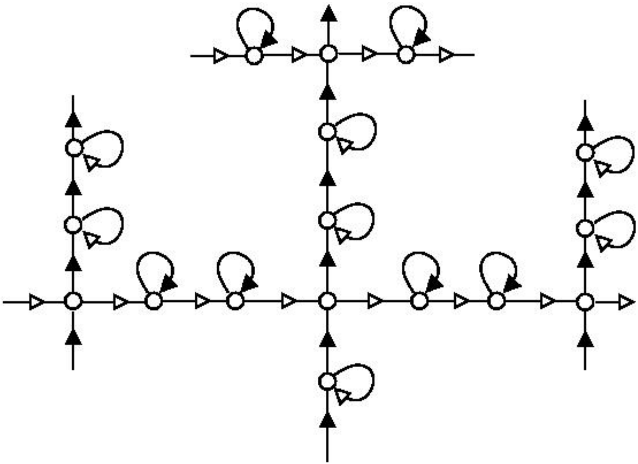
Theorem

The finitely-supported measures in $\mathcal{N}$ are dense and these correspond to normal subgroups of $\mathbb{F}_2$. Therefore,

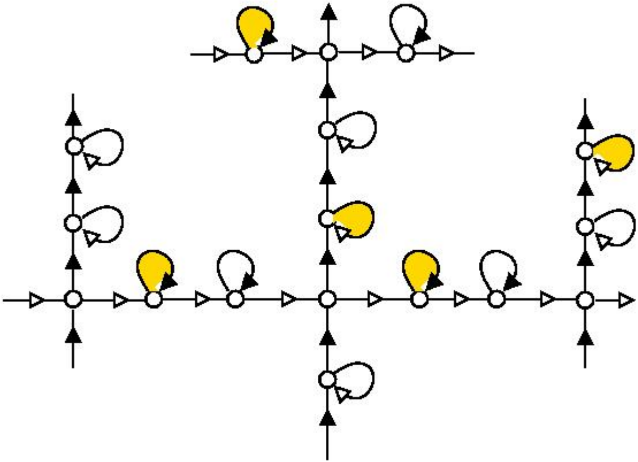
$$\{h_\mu(G) : \ G \textit{ a 2-generator group}\}$$

is dense in $[0, h_\mu(\mathbb{F}_2)]$.



Let $K_n < \mathbb{F}_2$ be the group generated by all elements of the form ghg^{-1} where $g \in \langle a^n, b^n \rangle$ and either $h = a^k b^r a^{-k}$ for some $1 \le |k| \le n-1$ and $r \in \mathbb{Z}$ or $h = b^k a^r b^{-k}$ for some $1 \le |k| \le n-1$ and $r \in \mathbb{Z}$.

A covering construction



Choose $0 \leq p \leq 1$ and choose each loop of $K_n \backslash \mathbb{F}_2$ with probability p independently.

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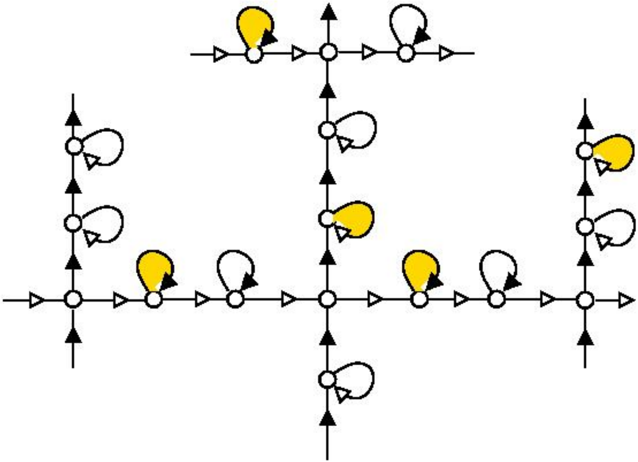

Choose $0 \leq p \leq 1$ and choose each loop of $K_n \backslash \mathbb{F}_2$ with probability p independently. Consider the resulting 2-complex.

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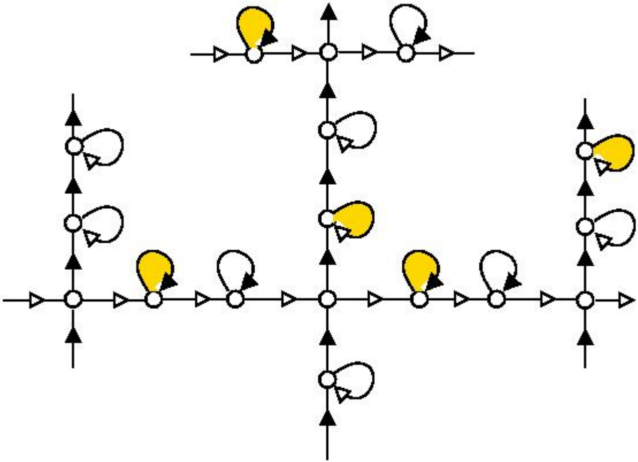

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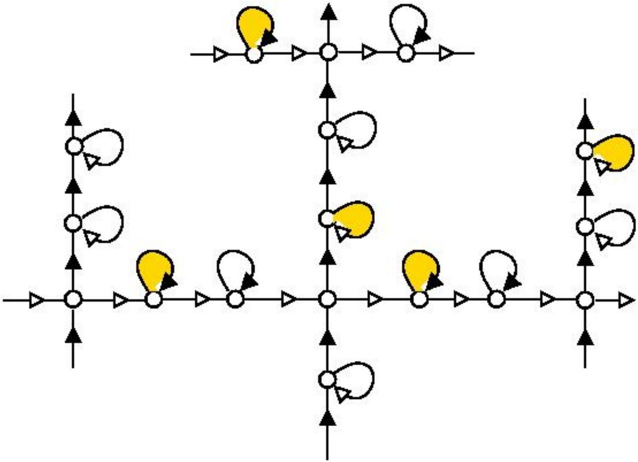

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Classification Results

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Theorem (Bader-Shalom, 2006)

If G_1, G_2 are just non-compact infinite property (T) groups then every ergodic IRS $K < G_1 \times G_2$ either splits as a product $K = H_1 \times H_2$ or K is a lattice subgroup a.s.

What sort of simplex is $M(G)$?

A simplex Σ is

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There are uncountably many nonisomorphic Bauer simplices.

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Let $M_{ie}(\mathbb{F}_r) := M_e(\mathbb{F}_r) \setminus M_{fi}(\mathbb{F}_r)$ and $M_i(\mathbb{F}_r) = \overline{\mathrm{Hull}(M_{ie}(\mathbb{F}_r))}$.

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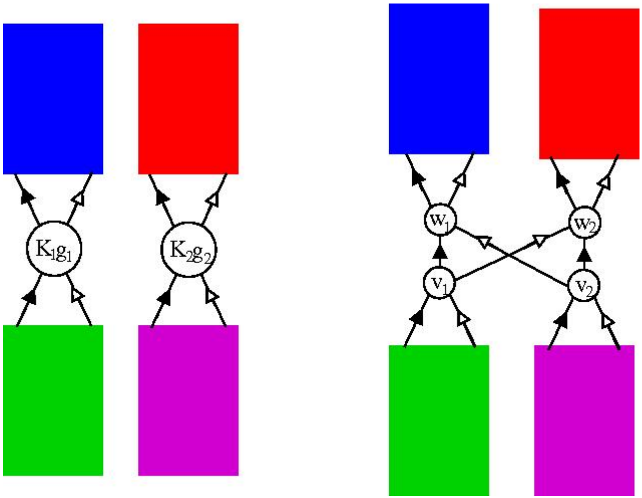
Theorem

$M_i(\mathbb{F}_r)$ is a Poulsen simplex. So $M_{ie}(\mathbb{F}_r) \cong L^2$.

Surgery



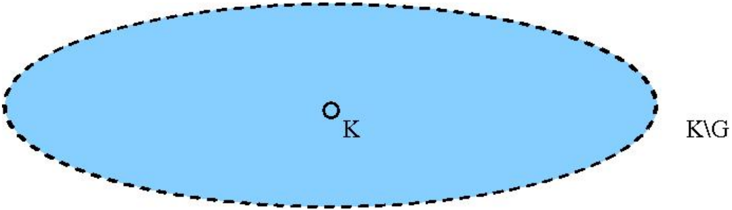
Given two Schreier coset graphs $K_1\backslash\mathbb{F}_r$, $K_2\backslash\mathbb{F}_r$, we can connect them together by replacing a vertex of each with 2 vertices and adding some edges.

Ergodic measures are dense

Let $\eta \in M_i(\mathbb{F}_r)$.

For $p \in (0,1)$ we will construct $\eta_p \in M_{ie}(\mathbb{F}_r)$ such that $\lim_{p \to 0} \eta_p = \eta$.

Building an ergodic approximation



Let $K < \mathbb{F}_r$ be random with law η.

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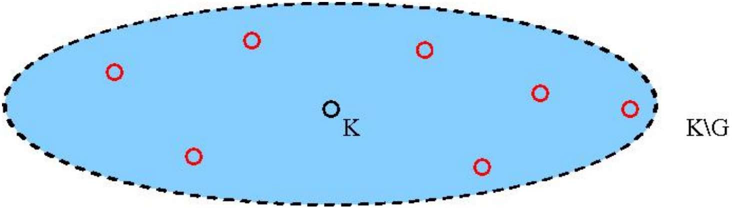
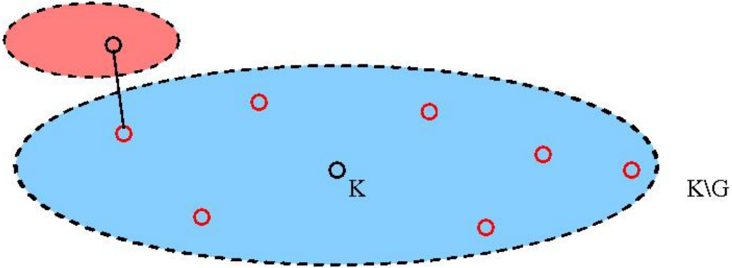


Color each vertex of $K\backslash\mathbb{F}_r$ red with prob. p independently.

Building an ergodic approximation



At a red vertex, choose a random subgroup $L < \mathbb{F}_r$ with law η independent of K and attach its Schreier coset graph by surgery to $K\backslash\mathbb{F}_r$.

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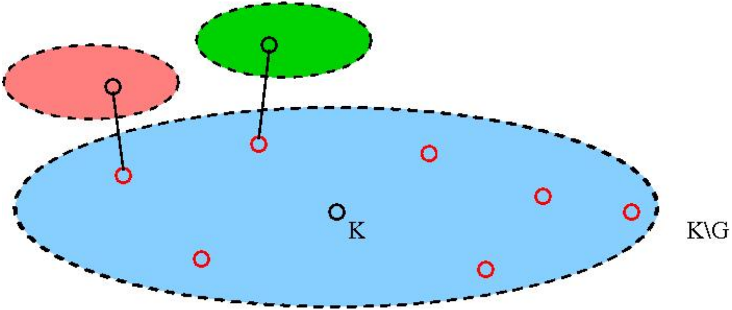

At a red vertex, choose a random subgroup $J < \mathbb{F}_r$ with law η independent of K and other subgroups and attach its Schreier coset graph by surgery to $K\backslash\mathbb{F}_r$.

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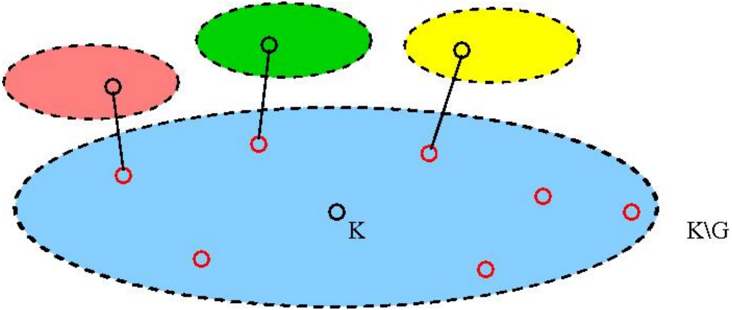

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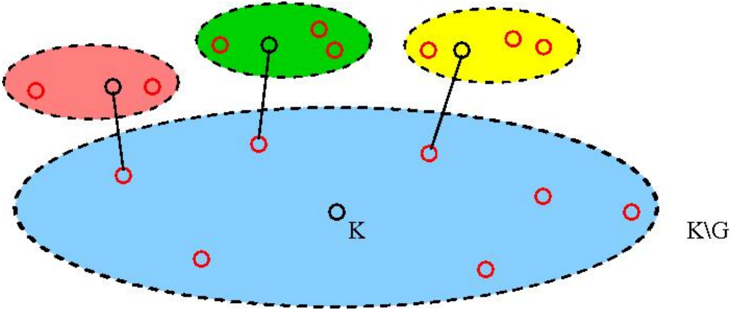

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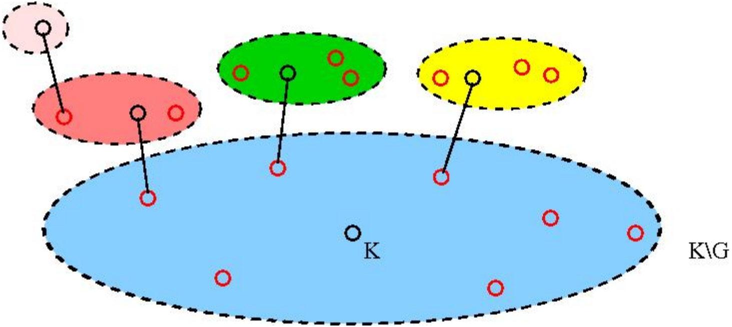

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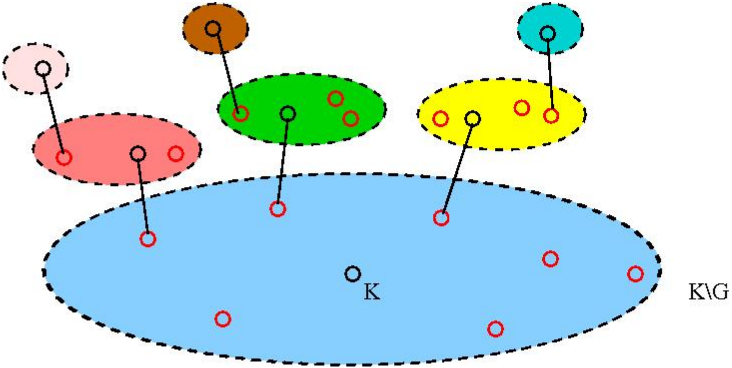

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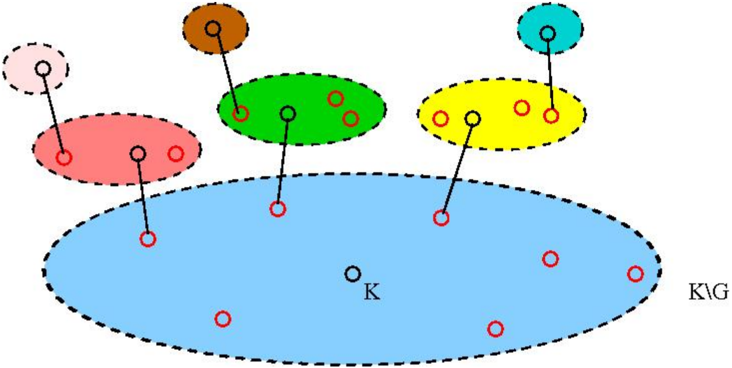

This is the Schreier coset graph of a random subgroup $K < \mathbb{F}_2$. Let η_p be the law of this subgroup. Show: η_p is ergodic and $\lim_{p\to 0} \eta_p = \eta$.

Further results and questions

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- (B.) Any ergodic aperiodic probability-measure-preserving equivalence relation (X, μ, E) with $\mathrm{cost}(E) < r$ is isomorphic to $(\mathrm{Sub}(\mathbb{F}_r), \lambda, E_{\mathbb{F}_r})$ for some $\lambda \in M(\mathbb{F}_r)$.

Further results and questions

- (Abert-Glasner-Weiss) If $K < G$ is an ergodic IRS then $\rho(K\backslash G) = \rho(G) \Leftrightarrow K$ is amenable a.s.
- (B.) Any ergodic aperiodic probability-measure-preserving equivalence relation (X, μ, E) with $\mathrm{cost}(E) < r$ is isomorphic to $(\mathrm{Sub}(\mathbb{F}_r), \lambda, E_{\mathbb{F}_r})$ for some $\lambda \in M(\mathbb{F}_r)$.
- (Bartholdi-Grigorchuk) There is a finitely generated group G with an ergodic IRS K so that the Schreier coset graph $K\backslash G$ has polynomial growth of irrational degree almost surely.