On topological full group of minimal homeomorphisms of a Cantor set

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A Cantor set

Theorem: A totally disconnected compact metric perfect space is homeomorphic to a Cantor set.

Different realizations:

(i) Space of sequences

A-finite alphabet, \( \{0, 1\} \)-binary alphabet

\[ \Omega = A^\mathbb{N}, \quad \Omega \ni w = w_1 w_2 \ldots w_n \ldots, \quad w_n \in A \]

or \( A^\mathbb{Z} \in \Omega = \ldots \eta_{-2} \eta_{-1} \eta_0 \eta_1 \ldots \), bi-infinite sequences
Tychonoff topology on $\mathcal{L}$ (⇔ topology of coordinatewise convergence).

$\tau : \mathcal{L} \to \mathcal{L}$, \( (\tau(\omega))_n = \omega_{n+1} \) - shift

(\(\mathcal{L}, \tau\)) - full shift

(\(X, \tau|_X\)) - subshift \( X \subseteq \mathcal{L} \) - closed, \( \tau \)-invariant subset

no isolated points in \(X\) \(\Rightarrow\) (\(X, \tau\)) - Cantor system.
(ii) Boundary of a tree

Cayley graph of $F_2$

$\partial T \approx \text{Cantor}$

$\partial T = \{ 0, 1 \}^\mathbb{N}$

$\alpha(w) = \begin{cases} 1, & \text{if } w \in 0 \cdot \mathbb{N} \\ 0 \alpha(w), & \text{if } w \in 1 \cdot \mathbb{N} \end{cases}$

$\alpha(0^n) = 1^n 0^n$ $\alpha(1^\infty) = 0^\infty$

(iii) Bratteli diagrams, Vershik transformations
Minimal Cantor Systems

\((X, T)\)

Cantor set \quad \text{homeomorphism}

Def. \((X, T)\) is \textit{minimal} if orbit

\[ \mathcal{O}(x) = \{ T^n(x) : n \in \mathbb{Z} \} \]

of each point \(x \in X\) is dense in \(X\)

\(\iff\) no proper non-empty \(T\)-invariant closed subsets

Example. (i) \((\mathbb{Z}_2, \text{odometer})\) - minimal system

(ii) \((\{0,1\}, \mathbb{Z})\) - not a minimal system (a lot of periodic points).
Example. Morse system \{0,1\}-alphabet

Blocks (words):

\[ B_0 = 0, \quad B_1 = B_0 \bar{B}_0 \]

\[ B_{n+1} = B_n \bar{B}_n, \]

where \( \bar{B}_n \) is the complement of \( B_n \) obtained by interchanging the 1's and 0's in \( B_n \)

\[ B_n < B_{n+1} \quad \text{prefix} \]

\[ \lim_{n \to \infty} B_n \quad \text{right infinite sequence} \]

Prouhet, Tupa, Morse

\[ 0^+ = 011010011001011001101001 \ldots \]
\{0,1\}^\mathbb{Z} \supset M = \{ \text{the set of sequences containing blocks that do appear in } x^+ \} \\
(M, \tau) - \text{ minimal system.}

The same example via substitution:

\[ B_n = \alpha^n(B_0) \]

\[ x^+ = \lim_{n \to \infty} B_n \]

Example

\[
\begin{align*}
0 & \rightarrow 0001 \\
1 & \rightarrow 0101
\end{align*}
\]

Substitutional dynamical system

Bratelli diagram, Vershik map
(iii) Toeplitz shifts

Def. A bi-infinite sequence \( x \in \mathbb{Z} \) is a Toeplitz sequence if the set of integers can be decomposed into arithmetic progressions such that entry \( x_i \) is constant on each arithmetic progression.

\[ \mathbb{Z} \ni x - \text{Toeplitz} \Rightarrow (X, \tau) - \text{minimal Cantor system} \]

\[ X = \overline{O_\tau(x)} - \text{closure of orbit} \]
(iv) Sturmian shifts

\[ \mathbb{T} = [0, 1) = \mathbb{R}/\mathbb{Z} = S^1 \]

\[ T_\alpha (x) = x + \alpha \pmod{1} \] (rotation by angle \( \alpha \), irrational)

\[ \mathcal{P} - \text{partition: } [0, 1) = [0, \alpha) \cup [\alpha, 1) \]

\[ [0, 1) \ni t \quad \longrightarrow \quad \text{itinerary} \quad x(t) = (x_i)_{i \in \mathbb{Z}} \in \{0, 1\}^\mathbb{Z} \]

\[ x_{\alpha} = \begin{cases} 0 & \text{if } T_\alpha^i(t) \in [0, \alpha) \\ 1 & \text{if } T_\alpha^i(t) \in [\alpha, 1) \end{cases} \]

\[ x = \{ x(t) : t \in [0, 1) \} \text{ - closure } (X, \tau) \text{ - minimal Cantor} \]
Topological entropy

\((X, \tau) = \text{subshift of } (A^\mathbb{Z}, \tau)\)

\[ h(X, \tau) = \lim_{h \to \infty} \frac{1}{n} \log |B_n(x)| \]

\(|B_n(x)| = \# \text{ of } n\text{-blocks appearing in points of } X\]

\(h\) is invariant of topological conjugacy

\[ f \circ T = S \circ f, \quad f - \text{homeomorphism}\]

and of flip conjugacy: \(T \sim S\) or \(T \sim S^{-1}\).
Topological Full Group (TFG) 
$(X,T)$ - Cantor system

Def. (i) Full group of $(X,T)$ is $[T]$ - the subgroup of all homeomorphisms $S \in \text{Homeo } X$ s.t.

$\forall x \in X$

$S(x) \in O(x) = \{ T^n(x) : n \in \mathbb{Z} \}$

$I \uparrow$

$S(x) = T^{n_S(x)}(x)$

$n_S : X \rightarrow \mathbb{Z}$ Borel measurable cocycle
(ii) \( G_T = \llbracket [T] \rrbracket \) - topological full group:

\[ G_T = \{ S \in [T] : n_S(x) - \text{continuous} \} \]

\( S \in G_T \iff \exists \text{ a finite clopen partition} \)

\{ C_1, \ldots, C_k \} \text{ of } X \text{ and a set of integers } \{ n_1, \ldots, n_k \}

\text{s.t.} \quad S\mid C_i = T^{n_i} \mid_{C_i} \quad \forall i = 1, \ldots, k
\[ \text{[T] is "huge", \([T]\) is countable} \]

Th. [Giordano, Putnam, Skau]. Let \((X,T)\) and \((Y,S)\) be Cantor minimal systems.

(i) They are orbit equivalent \(\iff [T] \cong [S]\)
(ii) They are flip conjugate \(\iff G_T \cong G_S\)

\[ \iff G_T^1 = G_S^1. \]

\[ T \sim S \text{ or } T \sim S^{-1} \]

Th. [GPS, Bezuglyi–Medynets, Matui].

1) \(G_T\) is indicable \((\exists \Psi: G_T \to \mathbb{Z})\)
2) The commutator subgroup $G_T^\prime$ is simple and if $N \triangleleft G_T$ then $G_T^\prime \leq N \leq G_T$.

3) $G_T^\prime$ is finitely generated $\iff (X,T)$ is topologically isomorphic to a minimal subshift over a finite alphabet.

4) $G_T^\prime$ is not finitely presented.

5) There is a normal subgroup $I \triangleleft G_T$ with $G_T/I \cong \mathbb{Z}$ and two locally finite subgroups $A, B \leq I$ s.t. $I = A \cdot B$. 
b). Any finite group can be embedded into $G_T$. Also $G_T$ contains $\mathbb{Z}_N \times \mathbb{Z}$. If $(X,T)$ is not odometer then Lamplighter group $\mathbb{Z}_L \otimes \mathbb{Z}$ embeds into $G^1_T$.


Th. [K. Juschenko, N. Monod]. The TFG $G_T$ of any minimal Cantor system is amenable.
The result.

Def. [A. Stepin, A. Vershik, 70-th] A group $G$ is LEF (locally embeddable into finite groups) if for every finite subset $F \subseteq G$ there is a finite group $H$ and a map $\psi: G \to H$ s.t.

(i) $\psi$ is injective on $F$

(ii) $\psi(gh) = \psi(g)\psi(h)$ $\forall g, h \in F$

[in the case $G$ is finitely generated this is equivalent to: $G$ is a limit of a sequence of finite groups in the space of marked groups. 1984]
Let and Amenable are "independent"

Sofic

Th. [Gri – Medynets] For any Cantor minimal system \((X, T)\) the topological full group \(G_T\) is LEF.

Cor. There are uncountably many finitely generated simple LEF groups. [as \(\forall h \geq 0\) there is a minimal Cantor subshift with topological entropy \(h\)].
On the proof.

\( A \subset X \)

\( \text{clopen} \)

\( t_A : A \to \mathbb{N} \)

\[ t_A(x) = \min \{ k \geq 1 : T^k(x) \in A \} \]

↑ function of the first return

\[ A_k = \{ x \in A : t_A(x) = k \} \]

\( k \in K = \text{Range}(t_A) \)

\[ A_k, T^{A_k}, \ldots, T^{k-1} A_k \quad \text{disjoint} \]

\[ A = \bigcup_{k \in K} A_k \]

\[ X = \bigcup_{k \in K} \bigcup_{i=0}^{k-1} T^i A_k \quad \text{partition} \]
Fix $x_0 \in X$

sequence $\{E_n\}_{n=1}^\infty$

clopen $\overline{E}_n$ - clopen partition

constructed by $\bigcap E_n$

Conditions:

1. $\{\overline{E}_n\}_{n \geq 1}$ generate the topology of $X$.

Kakutani-Rokhlin partition $\overline{\Xi}_n$
(2) \( \Xi_{n+1} \) refines \( \Xi_n \)

(3) \( \bigcap_n B(\Xi_n) = \{ x_0 \} \)

\( \Xi_n = \{ T^i B_v^{(n)} : 0 \leq i \leq h_v^{(n)} - 1, v \in V_n \} \)

\( V_n = \{ v_1, v_2, \ldots, v_n \} \)

Fix \( \{ m_n \}_{n=1}^{\infty} \), \( m_n \to \infty \) as \( n \to \infty \). Take a subsequence of \( \{ \Xi_n \}_{n=1}^{\infty} \) so that additionally

(4) \( h_n \geq 2m_n + 2 \), where \( h_n = \min_{v \in V_n} h_v^{(n)} \)
(5) The sets $T^i B(\Xi_n)$ have the property

$$\text{diam} \left( T^i B(\Xi_n) \right) < \frac{1}{n} \quad \text{for} \quad -m_n \leq i \leq m_n$$

Remark. (1)–(4) do not need minimality of $T$ (aperiodicity is enough). (5) holds only for minimal systems.
Def. Fix $n \geq 1$. $P \in G_T$ is $n$-permutation if

(i) its orbit cocycle $n_P(x)$ is compatible with the partition

(ii) $\forall x \in T_i B_v^{(n)} \quad (0 \leq i \leq h_v^{(n)} - 1, \quad v \in V_n)$

$0 \leq n_P(x) + i \leq h_v^{(n)} - 1$

[i.e. atoms of partition $\Xi_n$ are permuted only within each tower]

Def. $TA(x) = \begin{cases} 
\text{ta}_A(x) & \text{if } x \in A \\
T_A(x) & \text{if } x \in A \\
x & \text{if } x \notin A
\end{cases}$

induced transformation
\[ 0 \leq i \leq m_n \]

\[ U(i) = \bigcup_{v \in V_n} T^{h_v - i - 1}_v B^{(n)}_v \]

\[ D(i) = \bigcup_{v \in V_n} T^i B^{(n)}_v \]

"strips" at distance \( i \) from top and bottom respectively
Def. \( R \in G_{\mathbb{T}} \) is called an \( n \)-rotation with the rotation number \( \leq r \) if there are subsets \( S_U, S_D \subset \{ 0, 1, \ldots, m_n \} \) s.t.

\[ R = \prod_{i \in S_U} (T_{U(i)}^{l_i}) \times \prod_{j \in S_D} (T_{D(j)}^{k_j}) \]

\( |l_i| \leq r \), \( |k_j| \leq r \)

\( \text{rot}(R, R_2) \leq \text{rot}(R_1) + \text{rot}(R_2) \)
Proposition. Let $Q \in G_T$. Then there exists $n_0 > 0$ s.t. for all $n \geq n_0$, the homeomorphism $Q$ can be represented as $Q = PR$, where $P$ is an $n$-permutation and $R$ is an $n$-rotation with rotation number not exceeding 1.

Furthermore, the permutation $P$ can be represented (in a unique way) as a product of permutations $P_1, \ldots, P_{\nu}$, meeting the following conditions:

(i) The permutation $P_\nu$ acts only within $T$-tower $\Sigma^{(n)}$. 

(i) \( P_v(i) = P_w(i) \) for all \( i \in [-m_n, m_n] \), \( v, w \in V_n \),

(ii) The map \( P_v \) induces a permutation of \( 0, 1, \ldots, h_v^{(n)} - 1 \) with the property that

\[
\forall i \quad d_v^{(n)}(P_v(i), i) \leq n_0
\]

(iv) The rotation \( R \) acts only on levels which are within the distance \( n_0 \) to the top or the bottom of the partition.
(v) \[ Q = P_1 R_1 = P_2 R_2 \Rightarrow P_1 = P_2, \ R_1 = R_2 \]

uniqueness of decomposition

(vi) For any finite subset \( \{Q_1, \ldots, Q_k\} \) of \( G_T \) there is \( n_1 > n_0 \) s.t in the decomposition \( Q_i = P_i R_i \) all permutations \( P_i \) are different.

Proof of theorem. \[ F \subset G_T \quad |F| < \infty \]

Find \( n \in \mathbb{N} \) s.t \( \forall Q \in F^2, \ Q = P_Q R_Q \)
\[ P_Q \neq P_Z \text{ for } Q, Z \in F^2, Q \neq Z \]

\[ \exists d \in \mathbb{N} \text{ s.t. all } n\text{-rotations } R_Q, Q \in F \text{ are supported by levels } [-d, d], n \gg 1 \]

Can choose \( n \in \mathbb{N} \text{ s.t. } \forall Q \in F \)

\[ S_{Q, v}^{\pm 1} (i) = S_{Q, w}^{\pm 1} (i) \quad \forall i \in [-d, d] \]

\[ S_Q = \prod_{v \in V_n} S_{Q, v} \]

\[ \Rightarrow S_z^{-1} R_Q S_z \text{ is an } n\text{-rotation}, \forall Z, Q \in F \]
$H = \text{group of all } n\text{-permutations}$

Define $\varphi : F^2 \rightarrow H$

$\varphi(Q) = \varphi(P_Q R_Q) := P_Q$

$\varphi(Q Z) = \varphi(P_Q R_Q P_Z R_Z)$

$= \varphi((P_Q P_Z) P_Z^{-1} R_Q R_Z P_Z)$

$n\text{-permutation}$

$n\text{-rotations}$

$\Rightarrow \varphi(Q Z) = P_Q P_Z = \varphi(Q) \varphi(Z)$
Cor. \( G_T' \) is not finitely presented.