Groups associated with polynomial iterations

Volodymyr Nekrashevych

April 12, 2012

Let X be a finite set.

Let X be a finite set. Let $(a_i)_{i=1}^n$ be a sequence of elements of the symmetric group S_X.

Let X be a finite set. Let $(a_i)_{i=1}^n$ be a sequence of elements of the symmetric group S_X. Draw an oriented 2-dim CW-complex with vertices X, where we represent every cycle $(x_1, x_2, \ldots, x_k)$ of every permutation a_i by a 2-cell.

Let X be a finite set. Let $(a_i)_{i=1}^n$ be a sequence of elements of the symmetric group S_X. Draw an oriented 2-dim CW-complex with vertices X, where we represent every cycle $(x_1, x_2, \ldots, x_k)$ of every permutation a_i by a 2-cell. Different cells do not have common edges.

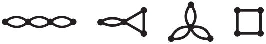
Let X be a finite set. Let $(a_i)_{i=1}^n$ be a sequence of elements of the symmetric group S_X. Draw an oriented 2-dim CW-complex with vertices X, where we represent every cycle $(x_1, x_2, \ldots, x_k)$ of every permutation a_i by a 2-cell. Different cells do not have common edges. We call it *cycle diagram* of the sequence.

Let X be a finite set. Let $(a_i)_{i=1}^n$ be a sequence of elements of the symmetric group S_X. Draw an oriented 2-dim CW-complex with vertices X, where we represent every cycle $(x_1, x_2, \ldots, x_k)$ of every permutation a_i by a 2-cell. Different cells do not have common edges. We call it *cycle diagram* of the sequence.

The sequence is *dendroid* if its cycle diagram is contractible.

$d = 2$ 

$d = 3$ 

$d = 4$ 

$d = 5$ 

If $a_1, a_2, \ldots, a_n$ is a dendroid sequence, then $a_1 a_2 \cdots a_n$ is a transitive cycle.

If $a_1, a_2, \ldots, a_n$ is a dendroid sequence, then $a_1 a_2 \cdots a_n$ is a transitive cycle.

More generally, for any partition of a dendroid sequence the sequence $(a_{i_{1,1}} \cdots a_{i_{1,k_1}}), (a_{i_{2,1}} \cdots a_{i_{2,k_2}}), \ldots, (a_{i_{m,1}} \cdots a_{i_{m,k_m}})$ is dendroid.

If $a_1, a_2, \ldots, a_n$ is a dendroid sequence, then $a_1 a_2 \cdots a_n$ is a transitive cycle.

More generally, for any partition of a dendroid sequence the sequence $(a_{i_{1,1}} \cdots a_{i_{1,k_1}}), (a_{i_{2,1}} \cdots a_{i_{2,k_2}}), \ldots, (a_{i_{m,1}} \cdots a_{i_{m,k_m}})$ is dendroid.

Suppose that $a_1, \ldots, a_n$ is dendroid. Let $b_i \in S_X$ be such that b_i is conjugate to a_i or a_i^{-1} in S_X, and $\langle b_1, \ldots, b_n \rangle$ is transitive.

If $a_1, a_2, \ldots, a_n$ is a dendroid sequence, then $a_1 a_2 \cdots a_n$ is a transitive cycle.

More generally, for any partition of a dendroid sequence the sequence $(a_{i_{1,1}} \cdots a_{i_{1,k_1}}), (a_{i_{2,1}} \cdots a_{i_{2,k_2}}), \ldots, (a_{i_{m,1}} \cdots a_{i_{m,k_m}})$ is dendroid.

Suppose that $a_1, \ldots, a_n$ is dendroid. Let $b_i \in S_X$ be such that b_i is conjugate to a_i or a_i^{-1} in S_X, and $\langle b_1, \ldots, b_n \rangle$ is transitive. Then $b_1, \ldots, b_n$ is dendroid.

Let T be a locally finite rooted tree.

Let T be a locally finite rooted tree. A set $g_1, \ldots, g_n$ of automorphisms of T is *dendroid* if it is a dendroid set of permutations of each level of T.

Let T be a locally finite rooted tree. A set $g_1, \ldots, g_n$ of automorphisms of T is *dendroid* if it is a dendroid set of permutations of each level of T.

The groups generated by dendroid sets of automorphisms of a rooted trees coincide with the *iterated monodromy groups of sequences of polynomials*.

Let f_i be a sequence of complex polynomials seen as a *backward iteration*

$$\mathbb{C} \xleftarrow{f_1} \mathbb{C} \xleftarrow{f_2} \mathbb{C} \xleftarrow{f_3} \cdots .$$

Suppose that this iteration is *post-critically finite*, i.e., there exists a finite set $M \subset \mathbb{C}$ such that M contains all critical values of $f_1 \circ f_2 \circ \cdots \circ f_n$ for every n.

Let f_i be a sequence of complex polynomials seen as a *backward iteration*

$$\mathbb{C} \xleftarrow{f_1} \mathbb{C} \xleftarrow{f_2} \mathbb{C} \xleftarrow{f_3} \cdots .$$

Suppose that this iteration is *post-critically finite*, i.e., there exists a finite set $M \subset \mathbb{C}$ such that M contains all critical values of $f_1 \circ f_2 \circ \cdots \circ f_n$ for every n.

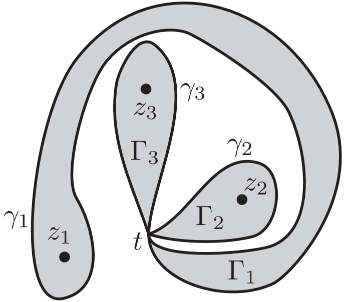
Then $\pi_1(\mathbb{C} \setminus M, t)$ acts on the tree of preimages

$$\bigsqcup_{n=0}^{\infty} (f_1 \circ f_2 \circ \cdots \circ f_n)^{-1}(t)$$

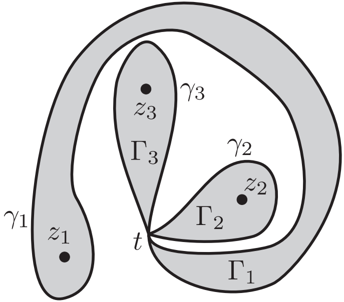
by the monodromy action.

Choose a *planar* generating set of $\pi_1(\mathbb{C} \setminus M, t)$.

Choose a *planar* generating set of $\pi_1(\mathbb{C} \setminus M, t)$.

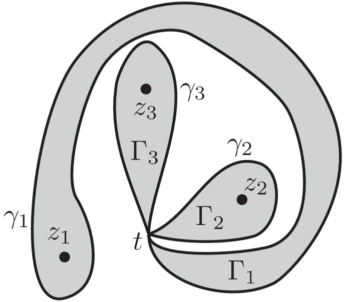


Choose a *planar* generating set of $\pi_1(\mathbb{C} \setminus M, t)$.



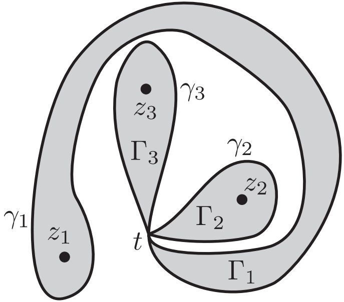
Then this set of generators is a dendroid set of automorphisms of the tree of preimages.

Choose a *planar* generating set of $\pi_1(\mathbb{C} \setminus M, t)$.



Then this set of generators is a dendroid set of automorphisms of the tree of preimages. The cycle diagrams are the preimages of the shaded area under $f_1 \circ \cdots \circ f_n$.

Choose a *planar* generating set of $\pi_1(\mathbb{C} \setminus M, t)$.



Then this set of generators is a dendroid set of automorphisms of the tree of preimages. The cycle diagrams are the preimages of the shaded area under $f_1 \circ \cdots \circ f_n$. Every dendroid set of automorphisms of a rooted tree can be obtained this way.

Examples

We can take one polynomial $f_i = f(z)$, if it is *post-critically finite*.

Examples

We can take one polynomial $f_i = f(z)$, if it is *post-critically finite*.
Examples: z^d, $z^2 - 1$, $T_d(z)$, $z^2 + i$, etc.

Examples

We can take one polynomial $f_i = f(z)$, if it is *post-critically finite*.
Examples: z^d, $z^2 - 1$, $T_d(z)$, $z^2 + i$, etc.

We can take z^2 and $1 - z^2$ in any order, since their critical values are 0 and 1, and the set $\{0, 1\}$ is invariant under both of them.

Examples

We can take one polynomial $f_i = f(z)$, if it is *post-critically finite*. Examples: z^d, $z^2 - 1$, $T_d(z)$, $z^2 + i$, etc.

We can take z^2 and $1 - z^2$ in any order, since their critical values are 0 and 1, and the set $\{0, 1\}$ is invariant under both of them. We get in this way an uncountable set of two-generated groups.

What can be said about the groups generated by dendroid sets of automorphisms of a rooted tree?

What can be said about the groups generated by dendroid sets of automorphisms of a rooted tree?

It follows from results of I. Benjamini, O. Shramm, A. Sharipa, and M. Abert, G. Elek that almost all connected components of the graph of the action of such a group on the boundary of the tree are amenable.

What can be said about the groups generated by dendroid sets of automorphisms of a rooted tree?

It follows from results of I. Benjamini, O. Shramm, A. Sharipa, and M. Abert, G. Elek that almost all connected components of the graph of the action of such a group on the boundary of the tree are amenable. In particular, the group can not have property (T).

What can be said about the groups generated by dendroid sets of automorphisms of a rooted tree?

It follows from results of I. Benjamini, O. Shramm, A. Sharipa, and M. Abert, G. Elek that almost all connected components of the graph of the action of such a group on the boundary of the tree are amenable. In particular, the group can not have property (T).

On the other hand, there exists a faithful action of $C_2 * C_2 * C_2 * C_2$ generated by a dendroid set of automorphisms of a rooted tree (constructed by M. Abert and G. Elek).

What can be said about the groups generated by dendroid sets of automorphisms of a rooted tree?

It follows from results of I. Benjamini, O. Shramm, A. Sharipa, and M. Abert, G. Elek that almost all connected components of the graph of the action of such a group on the boundary of the tree are amenable. In particular, the group can not have property (T).

On the other hand, there exists a faithful action of $C_2 * C_2 * C_2 * C_2$ generated by a dendroid set of automorphisms of a rooted tree (constructed by M. Abert and G. Elek). The graph of the action of this group on a generic orbit on the boundary coincides with the Cayley graph.

Automata

An *automaton* over an alphabet X consists of a set of *internal states* A and a map

$$\tau : A \times X \longrightarrow X \times A.$$

Automata

An *automaton* over an alphabet X consists of a set of *internal states* A and a map

$$\tau : A \times X \longrightarrow X \times A.$$

Given $a_1 \in A$ and a word $x_1 \ldots x_n \in X^*$, we compute $\tau(a_1, x_1) = (y_1, a_2)$, then $\tau(a_2, x_2) = (y_2, a_3)$, ..., $\tau(a_n, x_n) = (y_n, a_{n+1})$.

Automata

An *automaton* over an alphabet X consists of a set of *internal states* A and a map

$$\tau : A \times X \longrightarrow X \times A.$$

Given $a_1 \in A$ and a word $x_1 \ldots x_n \in X^*$, we compute $\tau(a_1, x_1) = (y_1, a_2)$, then $\tau(a_2, x_2) = (y_2, a_3)$, ..., $\tau(a_n, x_n) = (y_n, a_{n+1})$. The obtained word $y_1 y_2 \ldots y_n$ is the *image* of $x_1 x_2 \ldots x_n$ under the action of a_1.

A generalization

Consider an automaton over an alphabet X given by its *input set* A_1, *output set* A_2 and a map

$$\tau : A_1 \times X \longrightarrow X \times A_2.$$

A generalization

Consider an automaton over an alphabet X given by its *input set* A_1, *output set* A_2 and a map

$$\tau : A_1 \times X \longrightarrow X \times A_2.$$

An automaton is a *group automaton* if for every $a \in A_1$ the transformation mapping x to the first coordinate of $\tau(a, x)$ is a permutation.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be automata over alphabets X_1 and X_2, respectively;

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be automata over alphabets X_1 and X_2, respectively; with input sets A_1 and A_2, and output sets A_2 and A_3, respectively.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be automata over alphabets X_1 and X_2, respectively; with input sets A_1 and A_2, and output sets A_2 and A_3, respectively. Then their *product* $\mathcal{A}_2 \otimes \mathcal{A}_1$ is an automaton over $X_1 \times X_2$ with input set A_1, output set A_3, defined by successive application of the automata $\mathcal{A}_2$ and $\mathcal{A}_1$:

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be automata over alphabets X_1 and X_2, respectively; with input sets A_1 and A_2, and output sets A_2 and A_3, respectively. Then their *product* $\mathcal{A}_2 \otimes \mathcal{A}_1$ is an automaton over $X_1 \times X_2$ with input set A_1, output set A_3, defined by successive application of the automata $\mathcal{A}_2$ and $\mathcal{A}_1$:

$$\tau(a_1, x_1x_2) = (y_1y_2, a_3),$$

where

$$(y_1, a_2) = \tau(a_1, x_1), \qquad (y_2, a_3) = \tau(a_2, x_2).$$

Suppose we have a sequence of automata $\mathcal{A}_1, \mathcal{A}_2, \ldots$ over alphabets $X_1, X_2, \ldots$.

Suppose we have a sequence of automata $\mathcal{A}_1, \mathcal{A}_2, \ldots$ over alphabets $X_1, X_2, \ldots$. Suppose that the input and output sets of $\mathcal{A}_i$ are A_i and A_{i+1}.

Suppose we have a sequence of automata $\mathcal{A}_1, \mathcal{A}_2, \ldots$ over alphabets $X_1, X_2, \ldots$. Suppose that the input and output sets of $\mathcal{A}_i$ are A_i and A_{i+1}. Then $\mathcal{A}_n \otimes \cdots \otimes \mathcal{A}_1$ is an automaton over $X_1 \times X_2 \times \cdots \times X_n$, and A_1 hence acts on each of these sets.

Suppose we have a sequence of automata $\mathcal{A}_1, \mathcal{A}_2, \ldots$ over alphabets $X_1, X_2, \ldots$. Suppose that the input and output sets of $\mathcal{A}_i$ are A_i and A_{i+1}. Then $\mathcal{A}_n \otimes \cdots \otimes \mathcal{A}_1$ is an automaton over $X_1 \times X_2 \times \cdots \times X_n$, and A_1 hence acts on each of these sets. It acts by automorphism of the naturally defined rooted tree $\bigsqcup_{n=0}^{\infty} X_1 \times \cdots \times X_n$, Suppose we have a sequence of automata $\mathcal{A}_1, \mathcal{A}_2, \ldots$ over alphabets $X_1, X_2, \ldots$. Suppose that the input and output sets of $\mathcal{A}_i$ are A_i and A_{i+1}. Then $\mathcal{A}_n \otimes \cdots \otimes \mathcal{A}_1$ is an automaton over $X_1 \times X_2 \times \cdots \times X_n$, and A_1 hence acts on each of these sets. It acts by automorphism of the naturally defined rooted tree $\bigsqcup_{n=0}^{\infty} X_1 \times \cdots \times X_n$, where a word $x_1 \ldots x_n$ is adjacent to $x_1 \ldots x_n x_{n+1}$.

Dendroid automata

A group automaton over an alphabet X, input and output sets A and B is *dendroid* if the following conditions hold.

Dendroid automata

A group automaton over an alphabet X, input and output sets A and B is *dendroid* if the following conditions hold.

1. The set of permutations defined by A on X is dendroid.

Dendroid automata

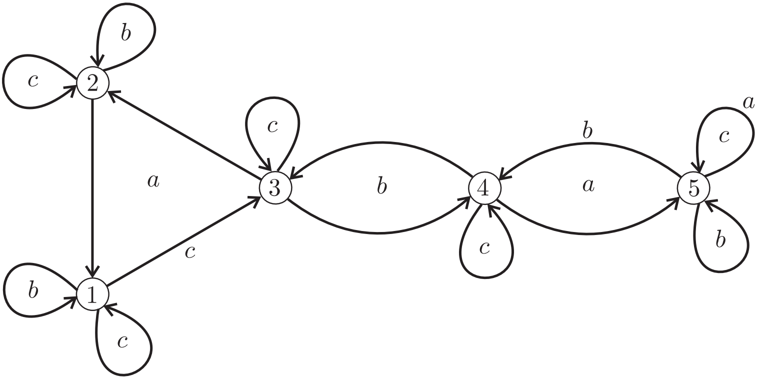
A group automaton over an alphabet X, input and output sets A and B is *dendroid* if the following conditions hold.

1. The set of permutations defined by A on X is dendroid.
2. For every $b \in B \setminus \{e\}$ there exists a unique pair $a \in A, x \in X$ such that $\tau(a, x) = (y, b)$ for some $y \in X$.

Dendroid automata

A group automaton over an alphabet X, input and output sets A and B is *dendroid* if the following conditions hold.

1. The set of permutations defined by A on X is dendroid.
2. For every $b \in B \setminus \{e\}$ there exists a unique pair $a \in A, x \in X$ such that $\tau(a, x) = (y, b)$ for some $y \in X$.
3. For any cycle $(x_1, x_2, \ldots, x_k)$ of the action of $a \in A$ on X we have $\tau(a, x_i) = (x_{i+1}, e)$ for all but possibly one index i. (Here indices are taken modulo k.)



b
c
2
a
c
3
b
4
b
a
a
c
5
b
c
b
1
c
c
b

Theorem

For any sequence $\mathcal{A}_1, \mathcal{A}_2, \dots$ of dendroid automata such that the input and output sets of $\mathcal{A}_i$ are A_i and A_{i+1}, the action of A_1 on the tree of words is dendroid.

Theorem

For any sequence $\mathcal{A}_1, \mathcal{A}_2, \dots$ of dendroid automata such that the input and output sets of $\mathcal{A}_i$ are A_i and A_{i+1}, the action of A_1 on the tree of words is dendroid.

For any dendroid set A of automorphisms of a rooted tree T there exists a sequence of dendroid automata as above such that the action of A_1 on the tree of words is conjugate to the action of A on T.

Theorem

Let $\mathcal{A}_i$ be a sequence of dendroid automata over alphabets X_i with input and output sets A_i and A_{i+1}.

Theorem

Let $\mathcal{A}_i$ be a sequence of dendroid automata over alphabets X_i with input and output sets A_i and A_{i+1}. Suppose that $|X_i|$ and $|A_i|$ are bounded.

Theorem

Let $\mathcal{A}_i$ be a sequence of dendroid automata over alphabets X_i with input and output sets A_i and A_{i+1}. Suppose that $|X_i|$ and $|A_i|$ are bounded. Then the group generated by A_1 does not contain free subgroups.

Theorem

Let $\mathcal{A}_i$ be a sequence of dendroid automata over alphabets X_i with input and output sets A_i and A_{i+1}. Suppose that $|X_i|$ and $|A_i|$ are bounded. Then the group generated by A_1 does not contain free subgroups.

Are such groups amenable?

Consider the following two dendroid automata $\mathcal{A}_0$ and $\mathcal{A}_1$ (over the alphabet $X = \{0, 1\}$ and input-output set $\{a, b, c, e\}$).

Consider the following two dendroid automata $\mathcal{A}_0$ and $\mathcal{A}_1$ (over the alphabet $X = \{0, 1\}$ and input-output set $\{a, b, c, e\}$).

$$\begin{cases} \tau(a,0) = (1,e), & \tau(a,1) = (0,e), \\ \tau(b,0) = (0,a), & \tau(b,1) = (1,c), \\ \tau(c,0) = (0,e), & \tau(c,1) = (1,b). \end{cases}$$

Consider the following two dendroid automata $\mathcal{A}_0$ and $\mathcal{A}_1$ (over the alphabet $X = \{0, 1\}$ and input-output set $\{a, b, c, e\}$).

$$\begin{cases} \tau(a,0) = (1,e), & \tau(a,1) = (0,e), \\ \tau(b,0) = (0,a), & \tau(b,1) = (1,c), \\ \tau(c,0) = (0,e), & \tau(c,1) = (1,b). \end{cases}$$

and

$$\begin{cases} \tau(a,0) = (1,e), & \tau(a,1) = (0,e), \\ \tau(b,0) = (0,a), & \tau(b,1) = (1,c), \\ \tau(c,0) = (0,b), & \tau(c,1) = (1,e). \end{cases}$$

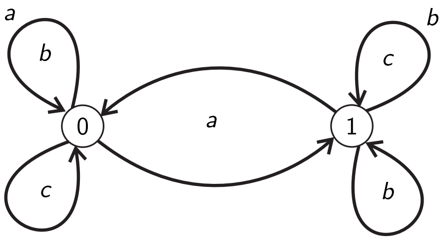
Consider the following two dendroid automata $\mathcal{A}_0$ and $\mathcal{A}_1$ (over the alphabet $X = \{0, 1\}$ and input-output set $\{a, b, c, e\}$).

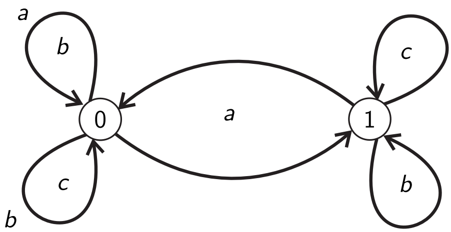
$$\begin{cases} \tau(a,0) = (1,e), & \tau(a,1) = (0,e), \\ \tau(b,0) = (0,a), & \tau(b,1) = (1,c), \\ \tau(c,0) = (0,e), & \tau(c,1) = (1,b). \end{cases}$$

and

$$\begin{cases} \tau(a,0) = (1,e), & \tau(a,1) = (0,e), \\ \tau(b,0) = (0,a), & \tau(b,1) = (1,c), \\ \tau(c,0) = (0,b), & \tau(c,1) = (1,e). \end{cases}$$

Choosing a sequence $w = i_1 i_2 \ldots \in \{0, 1\}^\infty$, we get a dendroid set of automorphisms of the binary rooted tree defined by the sequence $\mathcal{A}_{i_1}, \mathcal{A}_{i_2}, \ldots$. Let G_w be the group generated by these automorphisms.

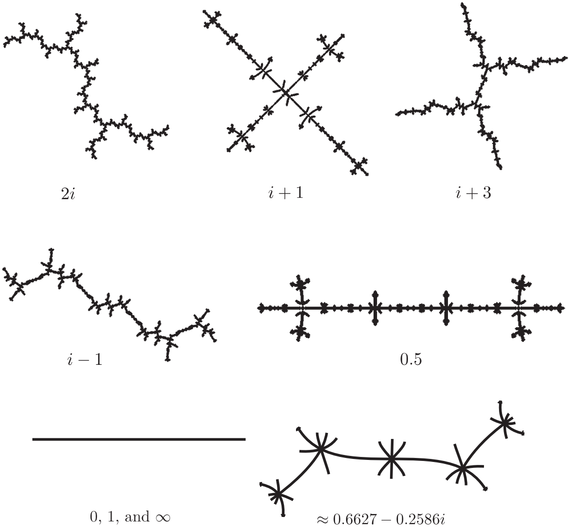

a
b
b
c
0
a
1
c
b

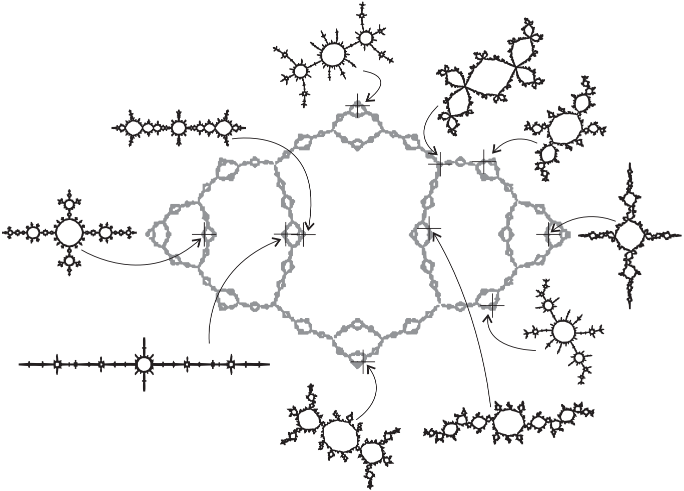

a
b
c
0
a
1
c
b
b

The groups G_w correspond to the sequences of polynomials of the form $f_i(z) = \left(1 - \frac{2z}{w_i}\right)^2$, where $w_i \in \mathbb{C}$ are such that $w_i = \left(1 - \frac{2}{w_{i+1}}\right)^2$.

The groups G_w correspond to the sequences of polynomials of the form $f_i(z) = \left(1 - \frac{2z}{w_i}\right)^2$, where $w_i \in \mathbb{C}$ are such that $w_i = \left(1 - \frac{2}{w_{i+1}}\right)^2$.

The Julia sets of analogous *forward* iterations resemble the graphs of action of the groups G_w on levels of the tree.


$2i$
$i+1$
$i+3$
$i-1$
0.5
0, 1, and ∞
$\approx 0.6627 - 0.2586i$



Two groups G_{w_1} and G_{w_2} are isomorphic if and only if w_1 and w_2 are cofinal.

Two groups G_{w_1} and G_{w_2} are isomorphic if and only if w_1 and w_2 are cofinal.

There exists uncountably many growth types in the family $\{G_w\}$.

Two groups G_{w_1} and G_{w_2} are isomorphic if and only if w_1 and w_2 are cofinal.

There exists uncountably many growth types in the family $\{G_w\}$.

The closure of G_w in the automorphism group of the tree does not depend on w.