Verbal subgroups of hyperbolic groups have infinite width

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Verbal set and verbal subgroup

Let $F(X)$ be a free group on a countable generating set $X = \{x_1, x_2, \dots\}$. Let $w \in F(X)$.

Let G be a group. $g \in G$ is called a w-element if g is an image of w under a homomorphism $F(X) \to G$.

One can think of w as a monomial $w = w(x_1, x_2, \dots, x_k)$. Then $w(g_1, g_2, \dots, g_k) \in G$ is the image of w under homomorphism extending map $x_i \to g_i$.

The set of w-elements in G, also called the set of values of w in G, is denoted $w[G]$:

$$\{g \in G \mid g = w(g_1, \ldots, g_k)\} = w[G].$$

The subgroup generated by $w[G]$ is denoted by $w(G)$:

$$\langle w[G] \rangle = w(G).$$

$w(G)$ is called w-verbal subgroup of G.

Examples:

- $w = x^{-1}y^{-1}xy$. $w(G) = [G, G]$.
- $w = x^2$, $G = \mathbb{Z}$. $w(G) = 2\mathbb{Z}$.
- $w = x^5y^{-2}$. $w(G) = w[G] = G$ since $g = g^5(g^2)^{-2} = w(g, g^2)$.

Represent w as

$$w = x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k} w',$$

where $w' \in [F, F]$. Denote $e(w) = \gcd(m_1, m_2, \dots, m_k)$, or $e(w) = 0$ if all $m_i = 0$.

If $e(w) = d > 0$, then every d-th power $g^d \in w[G]$. Indeed, there are $d_1, \dots, d_k$ such that

$$d_1 m_1 + \ldots + d_k m_k = d.$$

Then $w(g^{d_1}, g^{d_2}, \dots, g^{d_k}) = g^d$.

In particular, if $e(w) = 1$, then $w[G] = G$.

Words $w \in F$ s.t. $w \neq 1$ in F and $e(w) \neq 1$ are called proper.

Question: can elements of $w(G)$ be represented as a product of bounded number of values of $w^{\pm 1}$?

For a $g \in w(G)$, define its w-width:

$$l_w(g) = \min\{n \mid g = g_1 g_2 \cdots g_n,\ g_i^{\pm 1} \in w[G]\}.$$

w-width of G is defined to be

$$l_w(G) = \sup\{l_w(g) \mid g \in w(G)\},$$

which is a non-negative integer or infinity. If $l_w(G) < \infty$ for any w, we say that G is verbally elliptic:

$$\exists l \ w(G) \subseteq w^{\pm 1}[G]^l$$

If $l_w(G) = \infty$ for any proper w, we say that G is verbally parabolic:

$$\forall l \ w(G) \nsubseteq w^{\pm 1}[G]^l$$

History

- Ore's Conjecture (1951): Commutator width of non-abelian finite simple groups is 1. Established by Liebeck, O'Brian, Shalev and Tiep (2010).

- Serre's Conjecture: If G is a finitely generated profinite group then every subgroup of finite index is open. Proved by Nikolov and Segal (2007). Proof based on establishing uniform bounds on verbal width in finite groups.

In infinite groups, study was initiated by P. Hall.

- Stroud (1960's): All finitely generated abelian-by-nilpotent groups G are verbally elliptic.

- Rhemtulla (1968): All free products (except for infinite dihedral group) are verbally parabolic.

- Merzlyakov (1967): All linear algebraic groups are VE.
- Romankov (1982): All f.g. virtually nilpotent and virtually polycyclic groups are VE.
- Grigorchuk (1996): Groups in a wide class of amalgamated free products and HNN-extensions are VP.
- Bardakov: Braid groups are VP (1992), HNN-extensions with proper associated subgroups and one relator groups with at least three generators are VP (1997).
- See also Dobrynina (2000).

Theorem. Every non-elementary hyperbolic group G is VP, i.e., every proper verbal subgroup of G has infinite width.

Rhemtulla's gap function

In 1968, Rhemtulla showed that w-verbal subgroups free products (with exception to infinite dihedral group) have infinite width for every proper w.

For simplicity, we briefly trace his proof in case of a free group $F(a, b)$.

Suppose $e(w) = d$ and $g = w(g_1, g_2, \ldots, g_k)$. Then g_i can be cut into pieces so that each piece occurs in g a number of times divisible by d (counting inverse occurrences as -1).

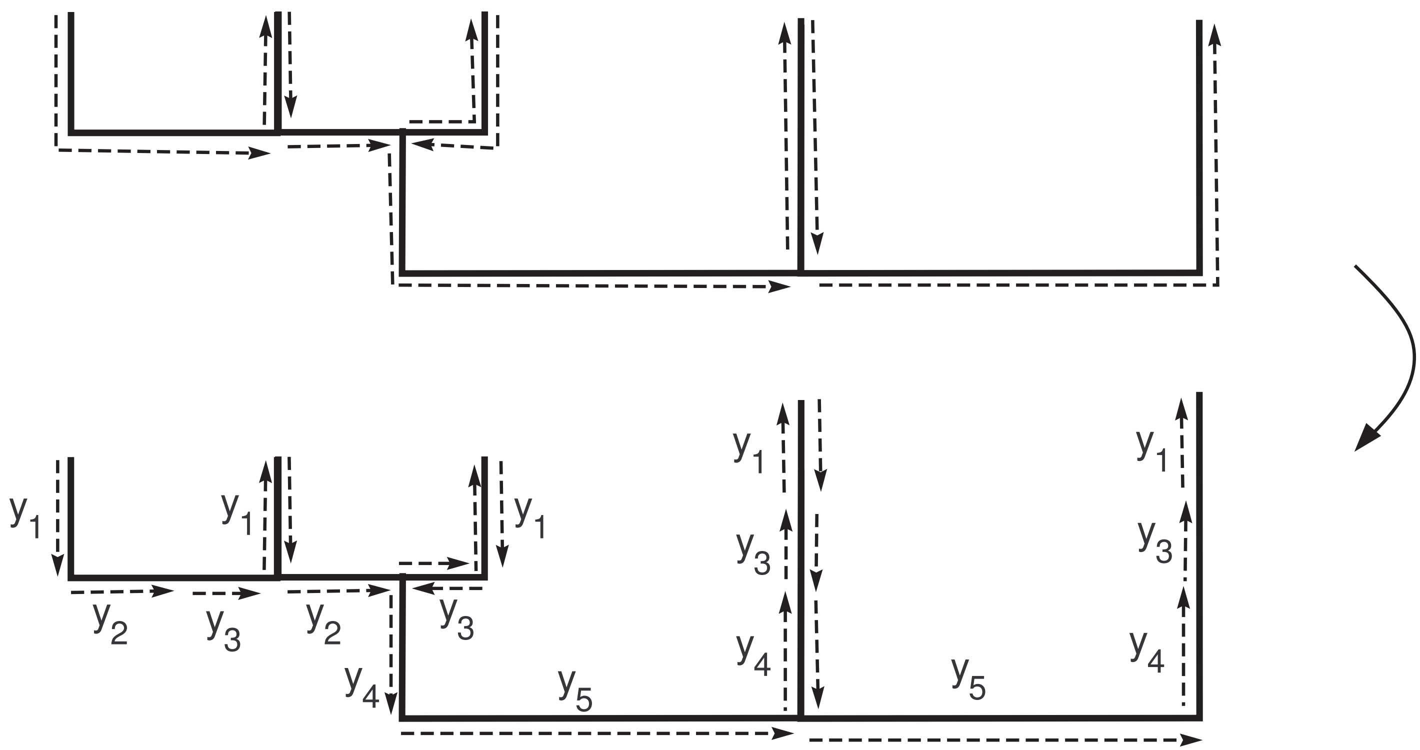

Figure 1: $w = x_1^2 x_2^2$.

So, if we count occurrences of a specific subword in g, we get $0 \bmod d$, except for subwords that "hit boundary between pieces".

The same holds if $g \in F(a,b)$ is a product of $\leq l$ values of $w^{\pm 1}$.

Specifically, Rhemtulla counts number of subwords of the form ba^jb: for all j, except for $L = L(w, l)$ values, number of occurrences of the subword ba^jb in $w_1w_2 \dots w_l$ is divisible by d.

In this context, a^j (or just j) is called a b-gap.

To disprove finite width, one can easily construct an element $g \in w(G)$ where arbitrarily many subwords of this form occur exactly 1 time. For example, in case $d > 1$, the following elements work:

$$g = (aba)^d (a^2ba^2)^d \dots (a^mba^m)^d.$$

Indeed, every subword of the form $ba^{2j+1}b$ $(j = 0, \dots, m-1)$ occurs exactly once.

Construction in the case $d = 0$ is more technically involved.

In other words, function $\gamma(g)$ that counts number of j's such that gaps a^j are "irregular", is bounded on $w^{\pm 1}[G]^l$ and unbounded on $w(G)$.

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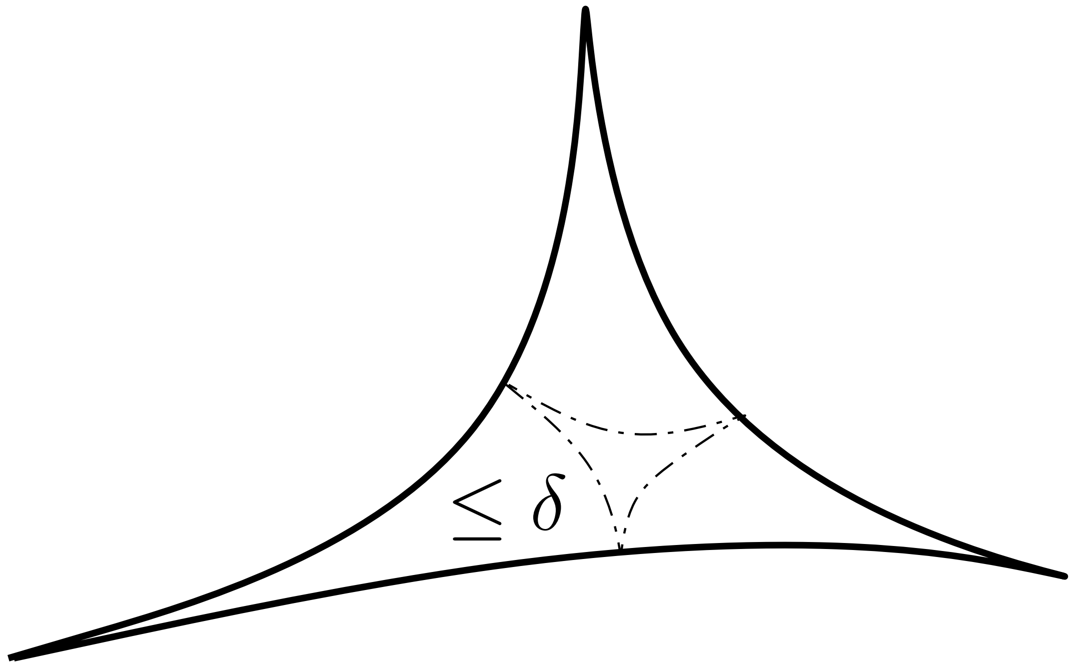
Alternative approach: adopt Fujiwara's treatment of second bounded cohomologies in hyperbolic groups.

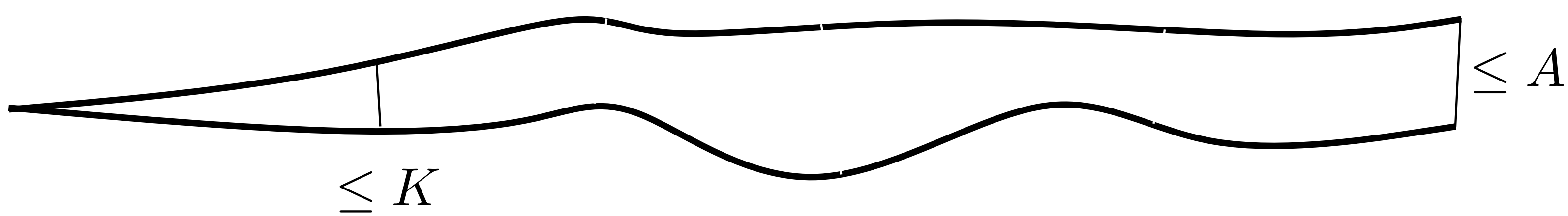
Hyperbolic spaces

Geodesic metric space is called δ-hyperbolic if all geodesic triangles are δ-thin:



δ-hyperbolic spaces possess fellow travel property:

if $p(t)$, $q(t)$ are two geodesic paths with $p(0) = q(0)$ and $|p(T) - q(T)| \leq A$, then there is a constant $K(\delta, A)$ s.t. $|p(t) - q(t)| \leq K$ for any $t \in [0, T]$.



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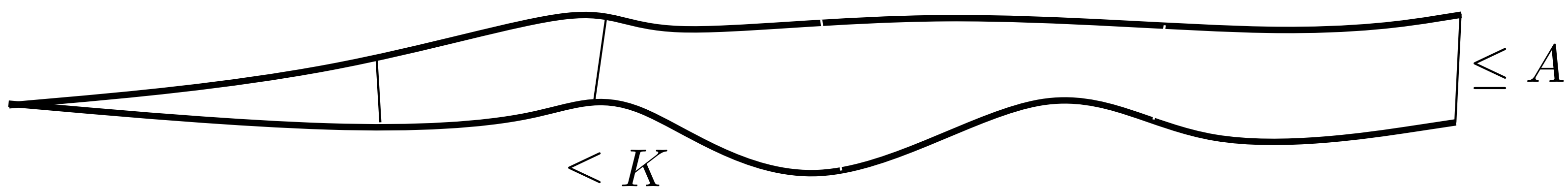

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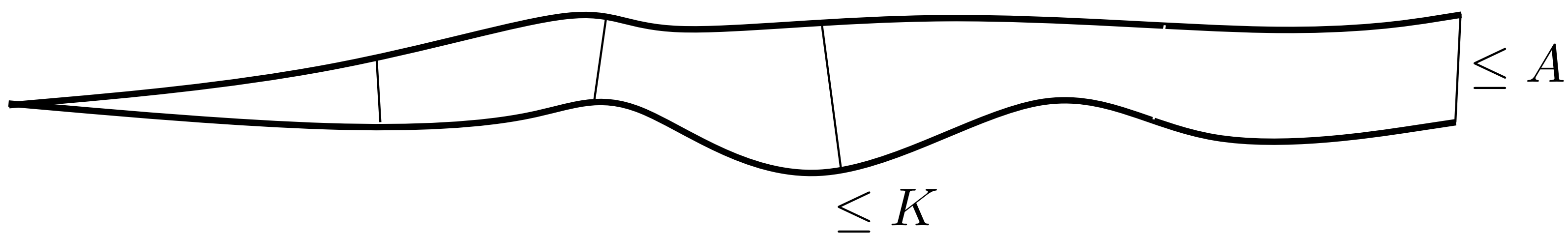

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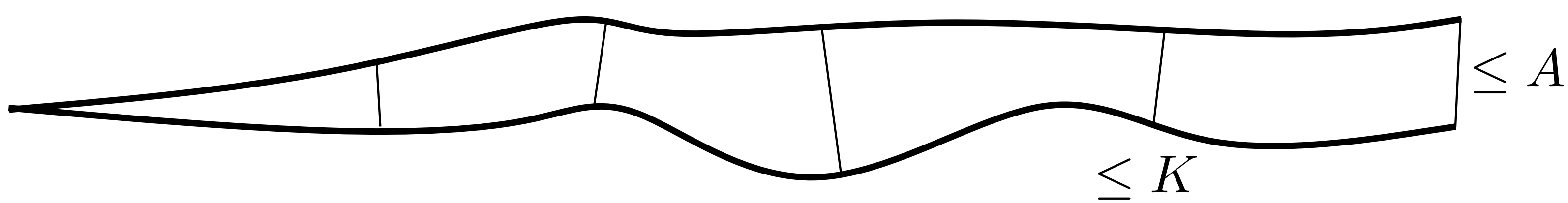


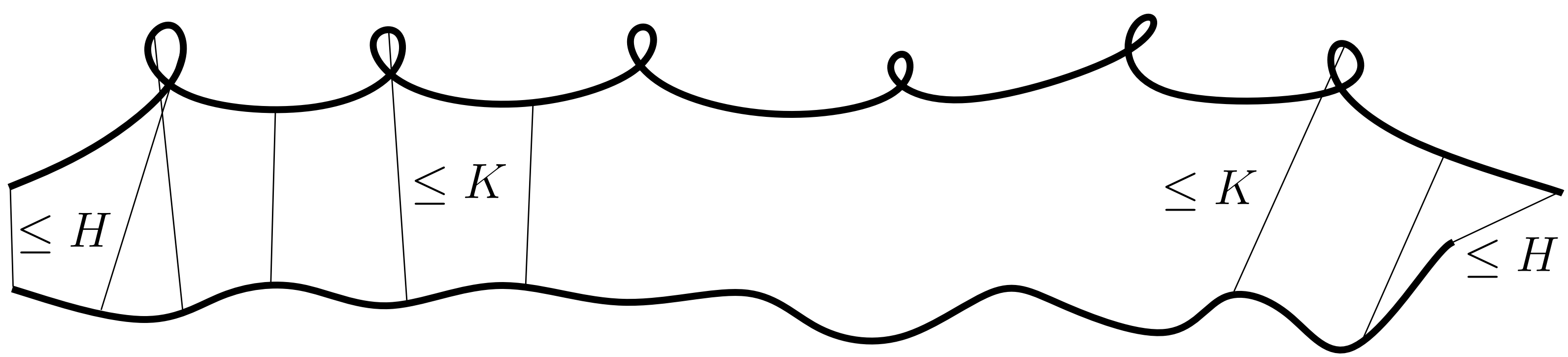
A path p in a metric space is called (λ, ε)-quasigeodesic if

$$\frac{1}{\lambda} \cdot |t - t'| - \varepsilon \leq |p(t) - p(t')| \leq \lambda \cdot |t - t'| + \varepsilon$$

We say that paths p, q asynchronously K-fellow travel if they possess monotone reparameterizations that K-fellow travel:

$$|p(\varphi(t)) - q(\psi(t))| \leq K.$$

Lemma. Let $\mathcal{H}$ be a δ-hyperbolic geodesic metric space. Let p, q be two (λ, ε)-quasigeodesic paths in $\mathcal{H}$ joining points P_1, P_2 and Q_1, Q_2, respectively. Suppose $H \geq 0$ is such that $|P_1Q_1| \leq H$ and $|P_2Q_2| \leq H$. Then there exists $K = K(\delta, \lambda, \varepsilon, H) \geq 0$ such that p, q asynchronously K-fellow travel.



Big Powers condition for hyperbolic groups

(Olshansky) Let $h_1, \ldots, h_\ell$ be elements infinite order in a hyperbolic group G such that $E(h_i) \neq E(h_j)$. Then there exists $N = N(h_1, \ldots, h_\ell) > 0$ such that

$$h_{i_1}^{m_1} h_{i_2}^{m_2} \cdots h_{i_s}^{m_s} \neq 1$$

whenever $i_k \neq i_{k+1}$ for $k = 1, \ldots, s-1$, and $|m_k| > N$ for $k = 2, \ldots, s-1$.

Moreover, the corresponding words are quasigeodesic with parameters that depend G and h_i, but not s.

Corollary. One can find elements b, f_0, f_1 such that if

$$g_1^{m_1} g_2^{m_2} \cdots g_k^{m_k} = {g'_1}^{m'_1} {g'_2}^{m'_2} \cdots {g'_l}^{m'_l},$$

where $g_i, g'_i \in D = \{b^{\pm 1}, f_0^{\pm 1}, f_1^{\pm 1}\}$, $m_i, m'_i > 0$, and $g_i \neq g_{i+1}^{\pm 1}$, $g'_i \neq (g'_{i+1})^{\pm 1}$, then $k = l$ and $g_i = g'_i$.

We fix appropriate b, f_0, f_1 and integer $M > 0$ (arises from certain technical reasons), and consider a set of elements $R = R(b, f_0, f_1, M) \subseteq G$ defined by

$$R = \{g \in G \mid \exists k \in \mathbb{N}, g_i \in D, m_i \geq M,\ g_{i-1} \neq g_i^{\pm 1},$$

$$g = g_1^{m_1} g_2^{m_2} \cdots g_k^{m_k}\}.$$

Defining gaps

For $g \in R$, its factor of the form

$$b^{m_\mu} g_{\mu+1}^{m_{\mu+1}} \cdots g_{\mu+\nu}^{m_{\mu+\nu}} b^{m_{\mu+\nu+1}},$$

where $g_i \neq b, b^{-1}$, is called a b-syllable. Define b^{-1}-syllables similarly.

With each b-syllable s we associate its b-gap, which is an integer $\omega_s \in \mathbb{Z}$ that counts number of occurrences of f_0 in s:

$$\omega_s = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_\nu,$$

where $\varepsilon_i = 0$ if $g_{\mu+i} = f_1^{\pm 1}$, and ε_i is such that $g_{\mu+i} = f_0^{\varepsilon_i}$, otherwise.

For $s = b^m \cdot f_0^{10} f_1^{100} f_0^{5} \cdot b^{m'}$, $\omega_s = 1 + 1 = 2.$

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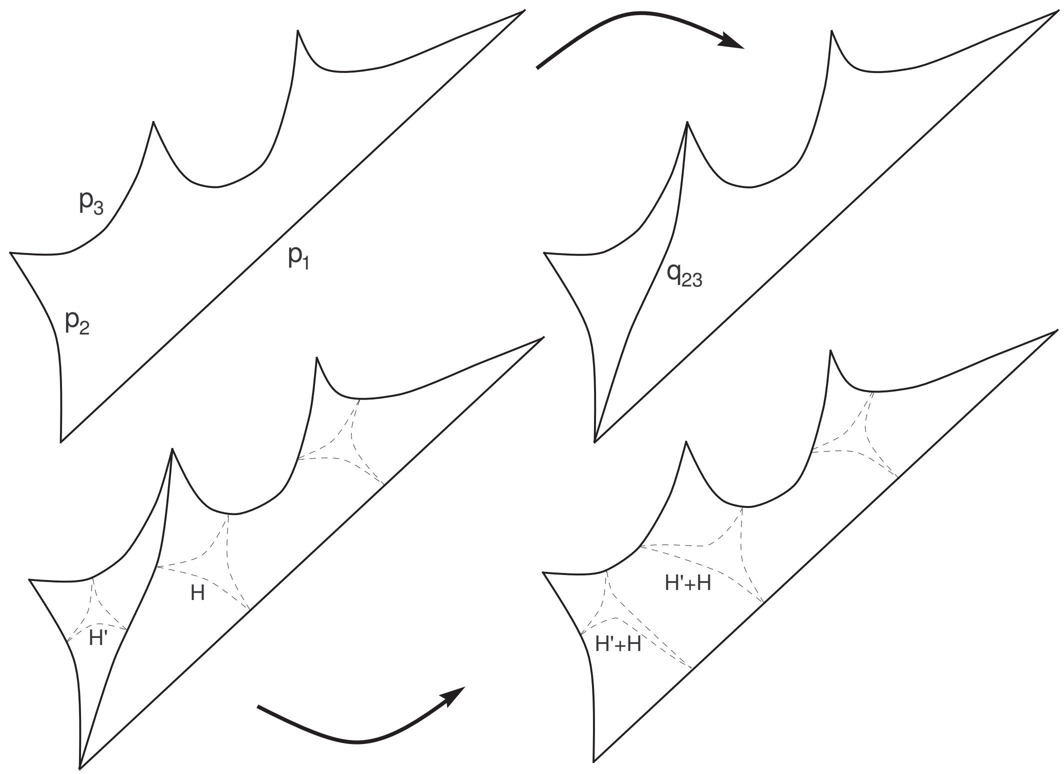
Function $\gamma : R \to \mathbb{Z}$ counts number of “irregular” gaps in g, that is the number of gaps that occur a number of times not divisible by $e(w)$.

We will show that γ is bounded on $R \cap w^{\pm 1}[G]^l$ and unbounded on $R \cap w(G)$.

(Note that if $xy \in R$, it does not guarantee that $x \in R$ and $y \in R$.)

Thin hyperbolic n-gons

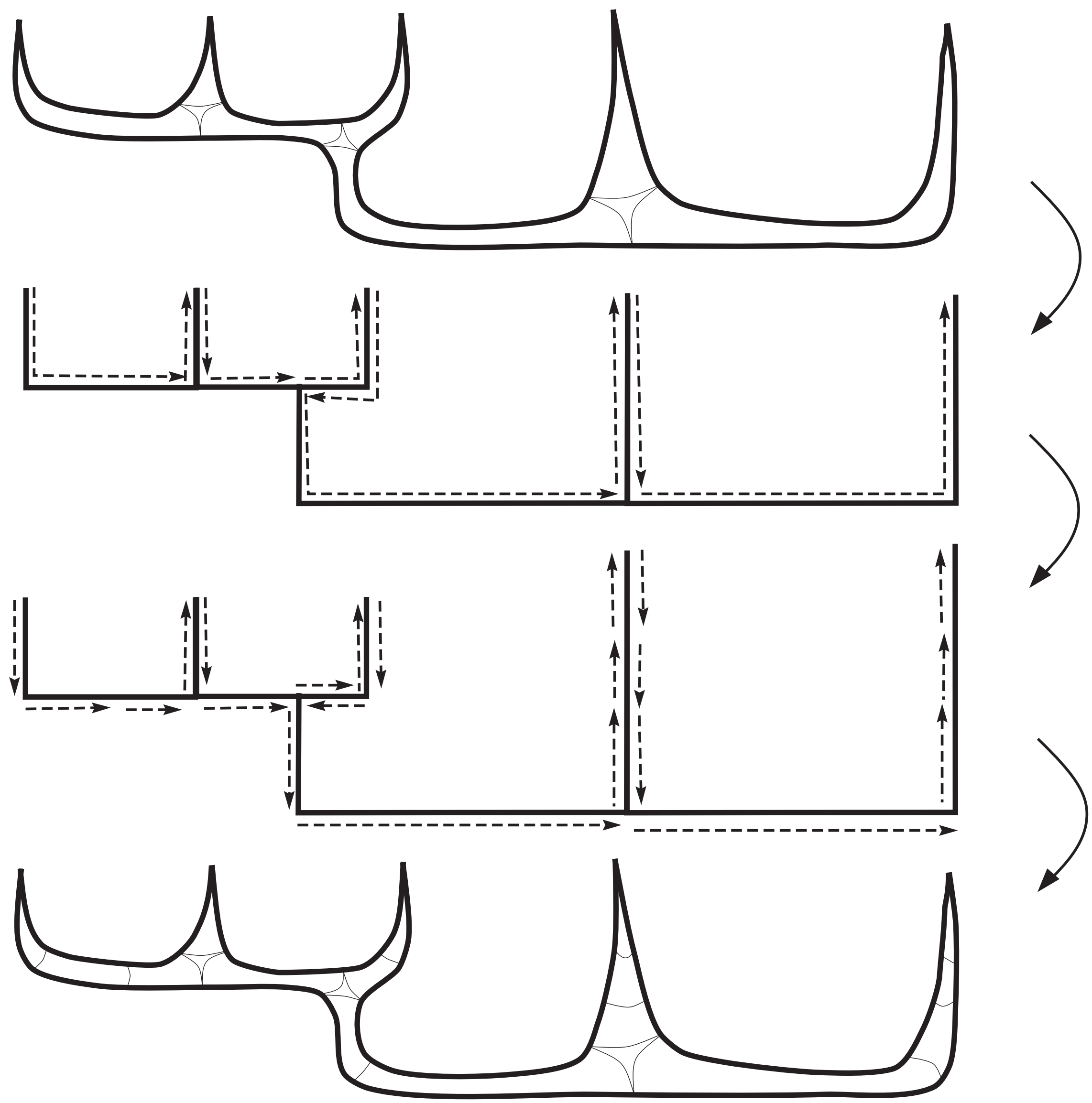
Since triangles in a hyperbolic space are δ-thin, all geodesic n-gons are also δ'-thin (where δ' depends on n):



By fellow travel property, the same holds for quasigeodesic n-gons (with a different δ' that depends on parameters of quasigeodesity).

This allows to "cut up" $g = w(g_1, \ldots, g_k)$ just as in case of free group (free product).

Suppose $g \in w[G] \cap R$. Let $w = x_{i_1} \ldots x_{i_N}$. Consider quasigeodesic $(N+1)$-gon whose sides are $g_{i_1}, \ldots, g_{i_N}$ and g^{-1}:



The big powers product g is therefore cut into pieces and each fellow-traveling class of pieces occurs (up to "short" artifacts on boundary) a number of times divisible by d (counting inverse occurrences as -1), therefore γ is bounded on $R \cap w[G]$.

It follows (considering longer word) that γ is bounded on $R \cap w^{\pm 1}[G]^l$.

It is easy to construct elements in $R \cap w(G)$ with arbitrarily large γ.

Indeed, for $d = e(w) > 1$, one can take basically the same example as in case of free groups:

$$X_j = (f_1^M f_0^M)^j b^M (f_1^M f_0^M)^j,$$

and

$$g = X_1^d X_2^d \dots X_m^d.$$

Case $d = 0$ is more technically involved.

Consequences

Observation: if a group G has a verbally parabolic homomorphic image, then G is verbally parabolic. Therefore, the following groups are VP (by original Rhemtulla's result):

- non-abelian residually free groups;
- pure braid groups (also follows from Bardakov's results);
- non-abelian right angled Artin gorups.

Consequence of the main result: non-elementary groups hyperbolic relative to proper residually finite subgroups (Osin) are VP. Thus, the following non-elementary groups are VP:

- the fundamental groups of complete finite volume manifolds of pinched negative curvature;
- $CAT(0)$ groups with isolated flats;
- groups acting freely on $\mathbb{R}^n$–trees.