The conjugacy problem in automaton groups is not solvable

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Outline

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Main result

Theorem (Sunic-V.)

There exist automaton groups (i.e. self-similar groups generated by finite self-similar sets) with unsolvable conjugacy problem.

Related results:

- Grigorchuk-Nekrashevych-Sushchanskiĭ (00): Is CP solvable for automaton groups ?
- WP is solvable for all such groups (straightforward, at most exponential time).
- WP is solvable in polynomial time, for the subclass of f.g. contracting groups.

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- Wilson-Zaleskii (97): CP for the Gupta-Sidki groups.
- Grigorchuk-Wilson (00): CP for all subgroups of finite index in the first Grigorchuk group.
- Bondarenko-Bondarenko-Sidki-Zapata (10): CP for groups generated by bounded automata (i.e. Pol(0) groups).
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Will use results from Bogopolski-Martino-Ventura:

Observation (B-M-V, 08)

Let H be f.g., and $\Gamma \leqslant \mathrm{Aut}(H)$ f.g. If $\Gamma \leqslant \mathrm{Aut}(H)$ is orbit undecidable then $H \rtimes \Gamma$ has unsolvable CP.

and

Proposition (B-M-V, 08)

For $d \geqslant 4$, there exist f.g., orbit undecidable, subgroups $\Gamma \leqslant \mathrm{GL}_d(\mathbb{Z})$.

and then show that

Theorem (Sunic-V.)

Let $\Gamma \leqslant \mathrm{GL}_d(\mathbb{Z})$ be f.g. Then, $\mathbb{Z}^d \rtimes \Gamma$ is an automaton group.

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For $d \geqslant 6$, $\mathrm{GL}_d(\mathbb{Z})$ contains f.g., orbit undecidable, free, subgroups.

Hence, we deduce:

Theorem (Sunic-V.)

For $d \geqslant 6$, there exists a f.p. group G simultaneously satisfying the following three conditions:

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Orbit decidability

(joint work with O. Bogopolski and A. Martino)

Definition

Let H be f.g. A subgroup $\Gamma \leqslant \mathrm{Aut}(H)$ is said to be orbit decidable (O.D.) if there is an algorithm s.t., given $u, v \in H$, it decides whether v and $\alpha(u)$ are conjugate, for some $\alpha \in \Gamma$.

First examples: $H = \mathbb{Z}^d$

Observation (folklore)

The full group $\mathrm{Aut}(\mathbb{Z}^d) = \mathrm{GL}_d(\mathbb{Z})$ is orbit decidable.

Proof. *For $u, v \in \mathbb{Z}^d$, there exists $A \in \mathrm{GL}_d(\mathbb{Z})$ such that $v = Au$ if and only if $gcd(u_1, \ldots, u_d) = gcd(v_1, \ldots, v_d)$.*

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For $A \in \mathrm{GL}_d(\mathbb{Z})$, the subgroup $\langle A \rangle \leqslant GL_d(\mathbb{Z})$ is O.D.

Proposition (Bogopolski-Martino-V., 08)

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OD subgroups in $\mathrm{Aut}(F_r)$

Examples over the free group: $H = F_r$

Theorem (Whitehead'30)

The full group $\mathrm{Aut}(F_r)$ is orbit decidable. That is, given $u, v \in F_r$ one can decide whether $v = \alpha(u)$ for some $\alpha \in \mathrm{Aut}(F_r)$.

Proof. *This is a classical and very influential result.*

Theorem (Brinkmann, 06)

Cyclic groups of $\mathrm{Aut}(F_r)$ are orbit decidable. That is, given $\varphi \in \mathrm{Aut}(F_r)$ and $u, v \in F_r$, one can decide whether $v = \varphi^n(u)$, up to conjugacy, for some $n \in \mathbb{Z}$.

Proof. *A difficult result using train-tracks.*

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Connection to semidirect products

Observation (B-M-V)

Let H be f.g., and $\Gamma \leqslant \mathrm{Aut}(H)$ f.g. If $H \rtimes \Gamma$ has solvable CP, then $\Gamma \leqslant \mathrm{Aut}(H)$ is orbit decidable.

Proof. *$G = H \rtimes \Gamma$ contains elements $(h, \gamma) \in H \times \Gamma$ operated like*

$$(h_1, \gamma_1) \cdot (h_2, \gamma_2) = (h_1\gamma_1(h_2), \gamma_1\gamma_2)$$

$$(h, \gamma)^{-1} = (\gamma^{-1}(h^{-1}), \gamma^{-1}).$$

For $h_1, h_2 \in H \leqslant G$, we have $h_1 \sim_G h_2 \;\Leftrightarrow\; \exists (h, \gamma) \in H \rtimes \Gamma$ s.t.

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Hence, $h_1 \sim_G h_2 \;\Leftrightarrow\; \exists \gamma \in \Gamma$ and $h \in H$ s.t. $h_1 = h\gamma(h_2)h^{-1}$. □

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In fact, for the free and free abelian cases (among others), the convers is also true, after “erasing the relations from Γ”:

Theorem (B-M-V, 08)

Let H be $\mathbb{Z}^d$ or F_r, and $\Gamma \leqslant \mathrm{Aut}(H)$ generated by $\alpha_1, \ldots, \alpha_m$. Then, $H \rtimes_{\alpha_1,\ldots,\alpha_m} F_m$ has solvable CP if and only if $\Gamma = \langle \alpha_1, \ldots, \alpha_m \rangle \leqslant \mathrm{Aut}(H)$ is orbit decidable.

Corollary

$\mathbb{Z}^d$-by-$\mathbb{Z}$ groups have solvable conjugacy problem.

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If $\Gamma = \langle M_1, \ldots, M_m \rangle$ is of finite index in $GL_d(\mathbb{Z})$ then $\mathbb{Z}^d \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

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Corollary (Bogopolski-Martino-Maslakova-V., 06)

Free-by-cyclic groups have solvable conjugacy problem.

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If $\Gamma = \langle \varphi_1, \ldots, \varphi_m \rangle$ has finite index in $\mathrm{Aut}(F_r)$ then $F_r \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

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Every F_2-by-free group has solvable conjugacy problem.

What we shall use is:

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Every F_2-by-free group has solvable conjugacy problem.

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Let H be f.g., and $\Gamma \leqslant \mathrm{Aut}(H)$ f.g. If $\Gamma \leqslant \mathrm{Aut}(H)$ is orbit undecidable then $H \rtimes \Gamma$ has unsolvable CP.

Connection to semidirect products

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Theorem (Miller, 70's)

There are free-by-free groups with unsolvable conjugacy problem.

So, there must be orbit undecidable subgroups in $\mathrm{Aut}(F_r)$, for $r \geqslant 3$. Where are them ?

Proposition (Bogopolski-Martino-V., 08)

Let H be a group, and let $A \leqslant B \leqslant \mathrm{Aut}(H)$ and $v \in H$ be such that $B \cap Stab^*(v) = 1$. Then,

$$OD(A) \text{ solvable} \quad \Rightarrow \quad MP(A, B) \text{ solvable.}$$

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So, deciding whether v can be mapped to w, up to conjugacy, by somebody in A, is the same as deciding whether φ belongs to A. Hence,

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Taking the copy B of $F_2 \times F_2$ in $\mathrm{Aut}(F_3)$ *via the embedding*

$$\begin{array}{rcl} F_2 \times F_2 & \hookrightarrow & Aut(F_3), \\ (u, v) & \mapsto & {}_u\theta_v : \begin{array}{rcl} F_3 & \to & F_3 \\ q & \mapsto & u^{-1}qv \\ a & \mapsto & a \\ b & \mapsto & b \end{array} \end{array}$$

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Proposition (B-M-V, 08)

For $d \geqslant 4$, there exist f.g., orbit undecidable, subgroups $\Gamma \leqslant GL_d(\mathbb{Z})$.

Proof. *Consider* $F_2 \simeq \langle P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \rangle \leq_{24} GL_2(\mathbb{Z})$.

- $Stab(1,0) = \{M \mid (1,0)M = (1,0)\} = \{\begin{pmatrix} 1 & 0 \\ n & \pm 1 \end{pmatrix} \mid n \in \mathbb{Z}\}$.
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- *Write* $v = (1, 0, 1, 0)$*. By construction,* $B \cap Stab(v) = \{I\}$*.*
- *Take* $A \leq B \simeq F_2 \times F_2$ *with unsolvable membership problem.*
- *By previous Proposition,* $A \leqslant \mathrm{GL}_4(\mathbb{Z})$ *is orbit undecidable.*
- *Similarly for* $A \leqslant \mathrm{GL}_d(\mathbb{Z})$*,* $d \geqslant 4$*.* □

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These orbit undecidable examples $\Gamma \leqslant GL_4(\mathbb{Z})$ come from Mihailova's construction, so they are not finitely presented...

Proposition (Sunic-V.)

For $d \geqslant 6$, $GL_d(\mathbb{Z})$ contains f.g., orbit undecidable, free, subgroups.

Proof. *Let $d \geqslant 6$.*

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- *Since $\langle s_1, \ldots, s_m\rangle \leqslant GL_2(\mathbb{Z})$ is free with basis $\{s_1, \ldots, s_m\}$, then ϕ must be one-to-one, and its image F is a free subgroup of $\mathrm{GL}_d(\mathbb{Z})$ or rank m.*
- *Easy to see that $F \leqslant \mathrm{GL}_d(\mathbb{Z})$ is orbit undecidable (using the orbit undecidability of $\langle g_1, \ldots, g_m\rangle = \Gamma \leqslant \mathrm{GL}_{d-2}(\mathbb{Z})$). $\square$*

In summary,

For $d \geqslant 6$, there exists a free $\Gamma \leqslant \mathrm{GL}_d(\mathbb{Z})$ such that $\mathbb{Z}^d \rtimes \Gamma$ has unsolvable CP.

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Outline

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(joint work with Z. Sunic)

Let X be an alphabet on k letters, and let X^* be the free monoid on X, thought as a rooted k-ary tree:

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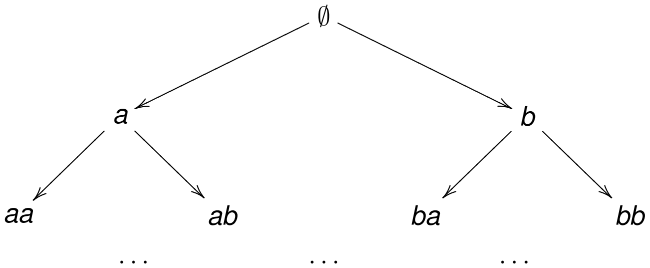

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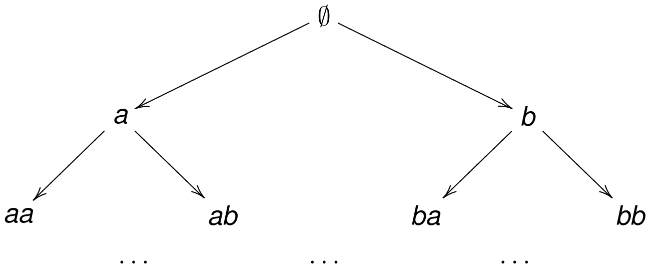

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Definition

- *A set of tree automorphisms is self-similar if it contains all sections of all of its elements.*
- *A finite automaton is a finite self-similar set (elements are called states).*
- *The group $G(\mathcal{A})$ of tree automorphisms generated by an automaton $\mathcal{A}$ is called an automaton group.*

The Grigorchuk group: $G = \langle \alpha, \beta, \gamma, \delta \rangle$, where

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Affinities of n-adic integers

Definition

Let $\mathcal{M} = \{M_1, \ldots, M_m\}$ be integral $d \times d$ matrices with non-zero determinants. Let $n \geqslant 2$ be relatively prime to all these determinants (thus, M_i is invertible over the ring $\mathbb{Z}_n$ of n-adic integers).

For an integral $d \times d$ matrix M and $\mathbf{v} \in \mathbb{Z}^d$, consider the invertible affine transformation ${}_{\mathbf{v}}M \colon \mathbb{Z}_n^d \to \mathbb{Z}_n^d$, ${}_{\mathbf{v}}M(\mathbf{u}) = \mathbf{v} + M\mathbf{u}$.

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$$G_{\mathcal{M},n} = \langle \{{}_{\mathbf{v}}M \mid M \in \mathcal{M},\ \mathbf{v} \in \mathbb{Z}^d\} \rangle \leqslant Aff_d(\mathbb{Z}_n).$$

Lemma

- *The group $G_{\mathcal{M},n}$ is finitely generated.*
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Proof. *Denote the translation by* $\tau_{\mathbf{v}} : \mathbb{Z}_n^d \to \mathbb{Z}_n^d$, $\mathbf{u} \mapsto \mathbf{u} + \mathbf{v}$.

Since ${}_{\mathbf{v}}M = \tau_{\mathbf{v}}\, {}_{\mathbf{0}}M$, *we have* $G_{\mathcal{M},n}$ *generated by* ${}_{\mathbf{0}}M$ *for* $M \in \mathcal{M}$, *and* $\tau_{\mathbf{e}_i}$, *where the* $\mathbf{e}_i$*'s are the canonical vectors.*

If $M \in \mathrm{GL}_d(\mathbb{Z})$, *then* ${}_{\mathbf{v}}M \in \mathit{Aff}_d(\mathbb{Z}_n)$ *restricts to an integral bijective affine transformation* ${}_{\mathbf{v}}M \in \mathit{Aff}_d(\mathbb{Z})$; *hence, we can view* $G_{\mathcal{M},n} \leqslant \mathit{Aff}_d(\mathbb{Z})$ *(and is independent from* n*; let's denote it by* $G_{\mathcal{M}}$*).*

They get multiplied as

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$G_{\mathcal{M}}$ is an automaton group

So, we have the groups $G_{\mathcal{M},n}$ (with $\mathcal{M} = \{M_1, \ldots, M_m\}$ as before) and

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Elements in $\mathbb{Z}_n$ may be (uniquely) represented as right infinite words over $Y_n = \{0, \dots, n-1\}$:

$$y_1 y_2 y_3 \cdots \quad \longleftrightarrow \quad y_1 + n \cdot y_2 + n^2 \cdot y_3 + \cdots .$$

Similarly, elements of $\mathbb{Z}_n^d$ (the free d-dimensional module, viewed as column vectors), may be (uniquely) represented as right infinite words over $X_n = Y_n^d = \{(y_1, \dots, y_d)^T \mid y_i \in Y_n\}$:

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Note that $|Y_n| = n$ and $|X_n| = n^d$.

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Definition

Construct the automaton $\mathcal{A}_{M,n}$:

- *Alphabet: X_n.*
- *States: $m_{\mathbf{v}}$ for $\mathbf{v} \in V_M$, with root permutation and sections*

$$m_{\mathbf{v}}(\mathbf{x}) = \mathrm{Mod}(\mathbf{v}+M\mathbf{x}), \quad \text{and} \quad m_{\mathbf{v}}|_{\mathbf{x}} = m_{\mathrm{Div}(\mathbf{v}+M\mathbf{x})}.$$

- *Straightforward to see that sections are again states.*

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Construct the automaton $\mathcal{A}_{M,n}$:

- *Alphabet: X_n.*
- *States: $m_{\mathbf{v}}$ for $\mathbf{v} \in V_M$, with root permutation and sections*

$$m_{\mathbf{v}}(\mathbf{x}) = \mathrm{Mod}(\mathbf{v} + M\mathbf{x}), \quad \text{and} \quad m_{\mathbf{v}}|_{\mathbf{x}} = m_{\mathrm{Div}(\mathbf{v}+M\mathbf{x})}.$$

- *Straightforward to see that sections are again states.*

$G_{\mathcal{M}}$ is an automaton group

$$_{\mathbf{v}}M(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\cdots) = \mathrm{Mod}(\mathbf{v} + M\mathbf{x}_1) + n \cdot_{\mathrm{Div}(\mathbf{v}+M\mathbf{x}_1)} M(\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4\cdots).$$

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Observation

The state $m_{\mathbf{v}} \in \mathcal{A}_{M,n}$ acts on a vector $\mathbf{u} = \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3 \cdots \in \mathbb{Z}_n^d$ as $m_{\mathbf{v}}(\mathbf{u}) = {}_{\mathbf{v}}M(\mathbf{u})$.

Definition

Construct the automaton $\mathcal{A}_{\mathcal{M},n}$ as the disjoint union of the automata $\mathcal{A}_{M_1,n}, \ldots, \mathcal{A}_{M_m,n}$.

- *Alphabet: X_n,*
- *It has $2^d \sum_{i=1}^{m} ||M_i||^d$ states.*

Proposition

$G_{\mathcal{M},n}$ is an automaton group generated by the automaton $\mathcal{A}_{\mathcal{M},n}$ (over an alphabet of size n^d, and having $2^d \sum_{i=1}^{m} ||M_i||^d$ states).

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$G_{\mathcal{M}}$ is an automaton group

Proof. *Clearly,* $G(\mathcal{A}_{\mathcal{M},n}) \leqslant G_{\mathcal{M},n}$.

For the other inclusion it remains to see that $\mathcal{A}_{\mathcal{M},n}$ *has enough states to generate* $G_{\mathcal{M},n}$. *In fact, for every* $M \in \mathcal{M}$, *we have states* $m_{\mathbf{0}}, m_{-\mathbf{e}_1}, \ldots, m_{-\mathbf{e}_d}$ *and so, also have*

$$m_{\mathbf{0}} = {}_{\mathbf{0}}M \colon \mathbf{u} \mapsto M\mathbf{u}$$

and

$$\tau_{\mathbf{e}_j} = m_{\mathbf{0}}(m_{-\mathbf{e}_j})^{-1} \colon \mathbf{u} \mapsto M^{-1}(\mathbf{e}_j + \mathbf{u}) \mapsto MM^{-1}(\mathbf{e}_j + \mathbf{u}) = \mathbf{e}_j + \mathbf{u},$$

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Conclusion

So, we have proved that

Theorem

For $d \geqslant 6$, there exists $\mathcal{M} = \{M_1, \ldots, M_m\}$ such that $\Gamma = \langle M_1, \ldots, M_m \rangle \leqslant \mathrm{GL}_d(\mathbb{Z})$ is free and orbit undecidable. Hence, the group $\mathcal{A}_{\mathcal{M},n} \simeq G_{\mathcal{M},n}$

- *is an automaton group,*
- *is $\mathbb{Z}^d$-by-free (i.e. $\simeq \mathbb{Z}^d \rtimes \Gamma$),*
- *has unsolvable conjugacy problem.*

THANKS