Leafages: how to prove the Hanna Neumann Conjecture.

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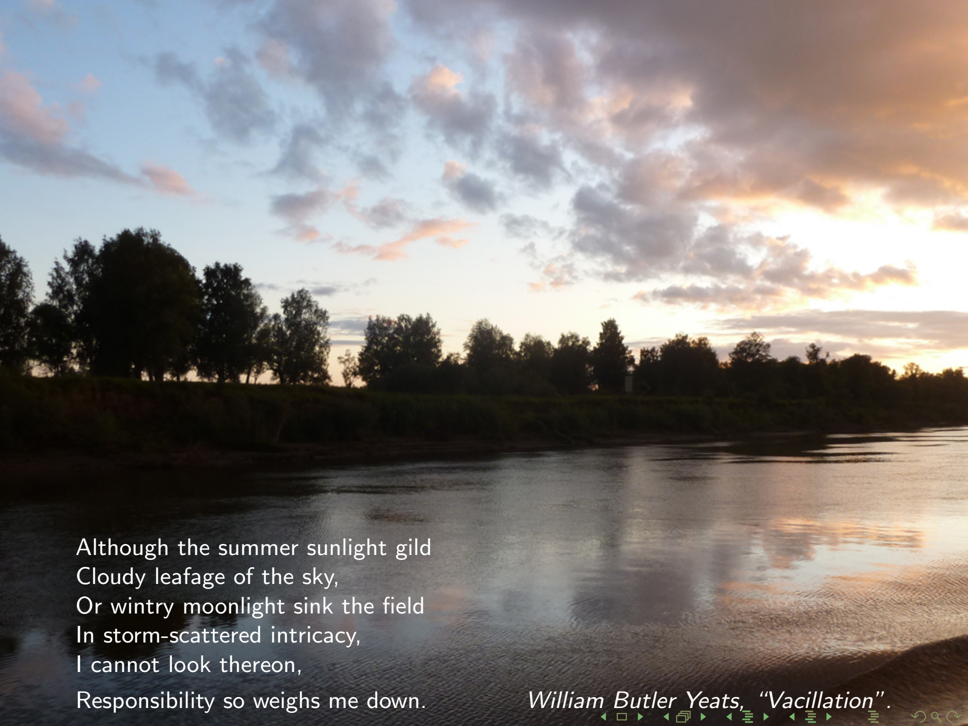
Although the summer sunlight gild
Cloudy leafage of the sky,
Or wintry moonlight sink the field
In storm-scattered intricacy,
I cannot look thereon,
Responsibility so weighs me down.
William Butler Yeats, "Vacillation".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Imagine you are in a *forest* ... Or in a *garden* ... Look around ...


You see *trees* and *flowers* around you … Look up again …

You see beautiful *leafage* above ...

Look closer ... You can now see *leaves* ... And *their veins* ...

We are now ready to start ...

Theorem (Hanna Neumann 1956-1957)

Suppose Γ *is a free group and* A *and* B *are its* *nontrivial* *finitely generated subgroups. Then*

$$\operatorname{rk}(A \cap B) - 1 \leq 2\,(\operatorname{rk} A - 1)(\operatorname{rk} B - 1).$$

Conjecture (HNC, Hanna Neumann 1956-1957)

Suppose Γ *is a free group and* A *and* B *are its* *nontrivial* *finitely generated subgroups. Then*

$$\operatorname{rk}(A \cap B) - 1 \leq (\operatorname{rk} A - 1)(\operatorname{rk} B - 1).$$

The *reduced rank* of a free group, due to Walter Neumann:

$$\bar{r}(A) := \max\{0, \operatorname{rk} A - 1\} \geq 0.$$

HNC restated

Suppose Γ *is a free group and* A *and* B *are its finitely generated subgroups, trivial or not. Then* $\bar{r}(A \cap B) \leq \bar{r}(A) \cdot \bar{r}(B)$.

Conjecture (SHNC, Walter Neumann 1989-1990)

Suppose Γ *is a free group and* A *and* B *are its finitely generated subgroups. Then*

$$\sum_{z \in s(A \backslash \Gamma / B)} \bar{r}(A^z \cap B) \leq \bar{r}(A) \cdot \bar{r}(B).$$

This talk is based on the following papers:

(1) *The topology and analysis of the Hanna Neumann Conjecture.*
The Journal of Topology and Analysis (JTA).
Proving HNC was not the goal of this paper. It rather uses analysis to **generalize the statement** SHNC, and gives **approaches to those generalizations**.

(2) *Submultiplicativity and the Hanna Neumann Conjecture.*
May 2011.
The proof of SHNC and of a more general result is given in the analytic language, it uses Hilbert modules and some graph theory.

(3) *Groups, graphs, and the Hanna Neumann Conjecture.*
May 2011.
The proof of the original SHNC purely in terms of groups and graphs.

Joel Friedman has also recently announced a proof of SHNC. Preprints:

(1) "A Proof of the Strengthened Hanna Neumann Conjecture", May 2009.
(2) "The Strengthened Hanna Neumann Conjecure I: A combinatorial proof", March 2010.
(3) "The Strengthened Hanna Neumann Conjecure I: A combinatorial proof (revised July 6, 2010)", July 2010.
(4) "Sheaves on Graphs and Their Homological Invariants", April 2011.
(5) "Sheaves on Graphs and a Proof of the Hanna Neumann Conjecture", April 2011.

Stallings' diagram:
fiber product of graphs

$$\begin{array}{ccc} S & \xrightarrow{\nu} & Z \\ \mu\downarrow & & \downarrow\beta \\ Y & \xrightarrow{\alpha} & X. \end{array}$$

$\pi_1(X) = \Gamma$,
$\pi_1(Y) = A$, $\pi_1(Z) = B$.

The *reduced rank* of a finite graph Y:

$$\bar{r}(Y) := \sum_{K \in Comp(Y)} \max\{0, -\chi(K)\}.$$

If Y is nonempty,

$$\bar{r}(Y) = \sum_{K \in Comp(Y)} \bar{r}(\pi_1(K)).$$

A restatement of SHNC: $\bar{r}(S) \overset{?}{\leq} \bar{r}(Y) \cdot \bar{r}(Z)$.
To prove SHNC, we want to describe $\bar{r}(S)$, $\bar{r}(Y)$, $\bar{r}(Z)$.

A tree

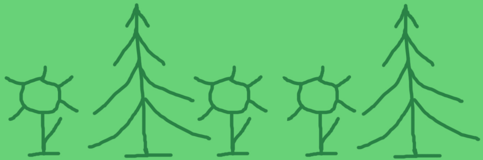
$\bar{r} = 0$

A flower

$\bar{r} = 0$

A forest

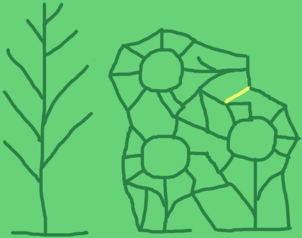
$\bar{r} = 0$



A garden

$\bar{r} = 0$

A graph Y.



An essential edge in Y.

An essential set in Y.

A maximal essential set in Y.

Formally:

- An *essential edge* in Y: removing it decreases reduced rank exactly by 1 (rather than by 0). *Inessential* otherwise.
- An *essential set* of edges, $E \subseteq E^Y$: one that decreases reduced rank exactly by $\#E$, i.e. $\bar{r}(Y \setminus E) = \bar{r}(Y) - \#E$.
- A *maximal essential set*, $E \subseteq E^Y$, realizes $\bar{r}(Y)$, i.e. $\bar{r}(Y \setminus E) = \bar{r}(Y) - \#E = 0$.

In other words, for a maximal essential set E, $\bar{r}(Y) = \#E$. This gives a description of reduced rank.

$$\begin{array}{ccc} S & \xrightarrow{\nu} & Z \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\beta} \\ Y & \xrightarrow{\alpha} & X. \end{array}$$

How can one try to prove HNC, and fail?

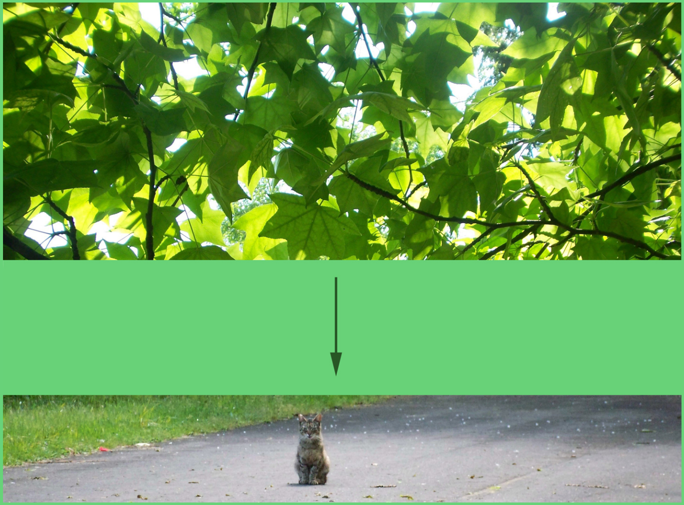
In many ways!

For example, one can pick a maximal essential set in S, then project it to Y and to Z.

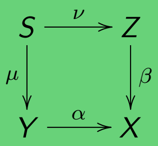
The problem is: those projections might not be be essential sets in Y or in Z. (Exercise: find a counterexample.)

Our plan is to make this idea work nevertheless.

A leafage is a map of cell complexes $\hat{S} \to \hat{Y}$ whose restriction to each component of $\hat{S}$ is injective.



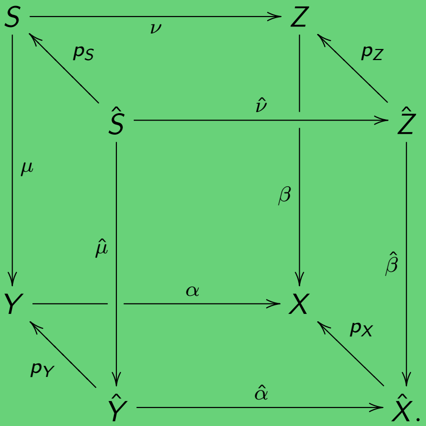
How do leafages arise?



- Let $\alpha : Y \to X$ and $\beta : Z \to X$ be *immersions* of complexes, defined as maps that can be extended to covers of X. The main example: the Stallings' immersions of finite graphs.
- Let S be their fiber-product.
- Let $\hat{X}$ be a complex with a free Γ-action whose quotient is X, for example the universal cover of X. In the main example, Γ is a free group.
- Let $p_X : \hat{X} \to X$ be the quotient map.

Pull this whole diagram back by p_X.

The result is a *system* of complexes:



Γ acts freely on $\hat{X}$, $\hat{Y}$, $\hat{Z}$, $\hat{S}$,

with quotients X, Y, Z, S.

If $\hat{X}$ is simply connected (connected or not), then $\hat{\alpha}$ and $\hat{\beta}$ are leafages. Hence $\hat{\mu}$ and $\hat{\nu}$ are leafages as well.

Now assume Γ is left-orderable. For example, any free group is left-orderable.

Define a Γ-invariant total order $\leq$ on $E^{\hat{X}}$:

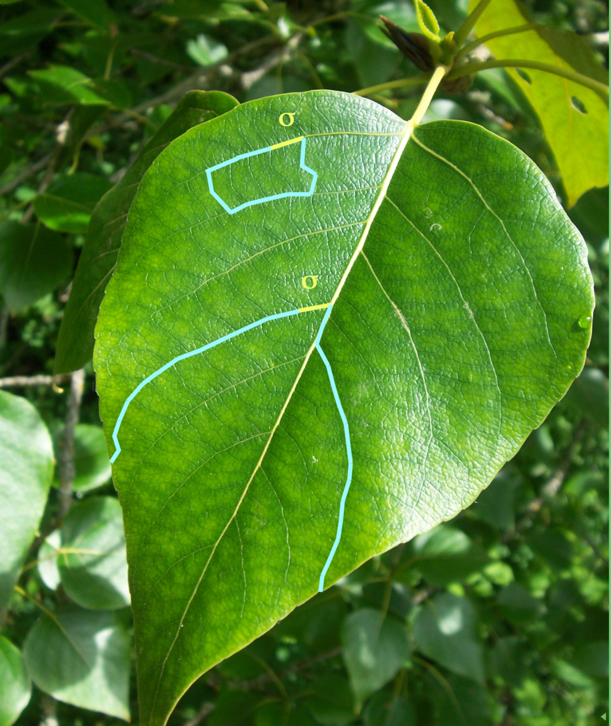
- Pick a Γ-transversal subset $\bar{E}^{\hat{X}}$ of $E^{\hat{X}}$. Then $E^{\hat{X}} \cong \Gamma \times \bar{E}^{\hat{X}}$.
- Put either lexicographic order on $E^{\hat{X}}$.

Put *pull-back orders* on $E^{\hat{Y}}$, $E^{\hat{Z}}$, $E^{\hat{S}}$. These are also Γ-invariant.

$$\begin{array}{ccc} \hat{S} & \xrightarrow{\hat{\nu}} & \hat{Z} \\ {\scriptstyle \hat{\mu}}\downarrow & & \downarrow{\scriptstyle \hat{\beta}} \\ \hat{Y} & \xrightarrow{\hat{\alpha}} & \hat{X} \end{array}$$

The restrictions of each leafage to each component is strictly order-preserving (on edges).

Now we define *veins* in a graph $\hat{Y}$.



Let $\sigma \in E^{\hat{Y}}$,
$E \subseteq E^{\hat{Y}} \setminus \{\sigma\}$.

$\leftarrow$ *A finite vein* at σ in E.

$\leftarrow$ *An infinite vein* at σ in E.

If $\hat{Y}$ is a forest, all veins are infinite.

We say that σ *falls into* E if there is a vein at σ in E.

The most important definition:
An edge σ in $\hat{Y}$ is *order-essential* if it falls into

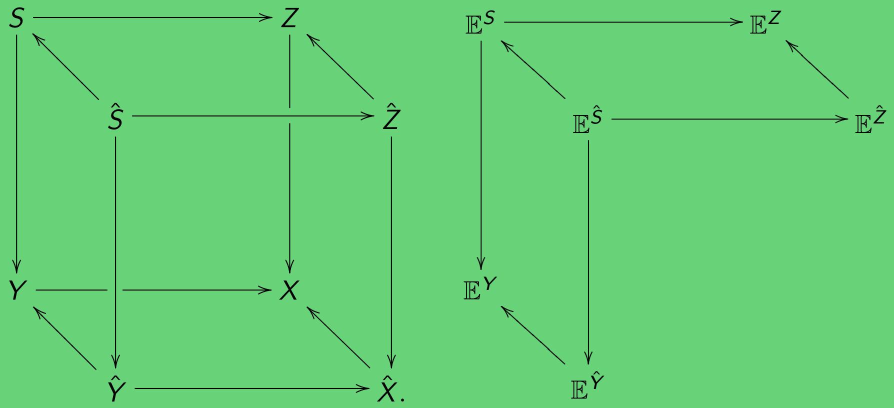
$$[E^{\hat{Y}} < \sigma] := \{\tau \in E^{\hat{Y}} \mid \tau < \sigma\}.$$

[An analytic comment: this is equivalent to $\partial\sigma \in \overline{\partial(\ell^2[E^{\hat{Y}} < \sigma])}$.]
Order-inessential otherwise.

$\mathbb{E}^{\hat{Y}}$:= the set of order-essential edges in $\hat{Y}$, $\mathbb{E}^{Y} := \Gamma\backslash\mathbb{E}^{\hat{Y}}$.
$\mathbb{I}^{\hat{Y}}$:= the set of order-inessential edges in $\hat{Y}$.

$\mathbb{E}^{\hat{Y}}$ and $\mathbb{I}^{\hat{Y}}$ are Γ-invariant.

The most important observation:
leafages map order-essential edges to order-essential edges.



Hence $\#\mathbb{E}^S \le \#\mathbb{E}^Y \cdot \#\mathbb{E}^Z$.
(Regardless of whether the graphs $\hat{Y}$, $\hat{Z}$, $\hat{S}$ are forests or not.)

It remains to show that $\#\mathbb{E}^Y = \bar{r}(Y)$.

To show that $\#\mathbb{E}^Y = \bar{r}(Y)$, it suffices to show that $\#\mathbb{E}^Y$ is a maximal essential set in Y.

(1) To show that $\#\mathbb{E}^Y$ is **essential** we need:

The deep-fall property for graphs

If $\sigma \in \mathbb{E}^{\hat{Y}}$ (i.e. if σ falls into $[E^{\hat{Y}} < \sigma]$), then σ falls into $[\mathbb{I}^{\hat{Y}} < \sigma]$.

Sketch of proof:

- Enumerate $[\mathbb{E}^{\hat{Y}} < \sigma]$ as $\{\sigma_1, \sigma_2, \ldots\}$.
- For each n, relabel $\{\sigma_1, \ldots, \sigma_n\}$ as $\{\tau_1, \ldots, \tau_n\}$ so that $\tau_1 > \ldots > \tau_n$.
- σ has a vein in $[E^{\hat{Y}} < \sigma]$.
- Each $\tau_i < \sigma$ and has a vein in $[E^{\hat{Y}} < \tau_i]$.
- Replace τ_i's with their veins one by one to construct a vein at σ that misses $\{\tau_1, \ldots, \tau_n\} = \{\sigma_1, \ldots, \sigma_n\}$.
- Then there is a limiting vein at sigma that misses $[\mathbb{E}^{\hat{Y}} < \sigma]$.

Then remove the edges of $\#\mathbb{E}^Y$ from Y one by one and check that reduced rank decreases exactly by 1 at each step. (Use the fact that $\hat{Y}$ is a forest.)

(2) To show that $\#\mathbb{E}^Y$ is **maximal** we need to show that $\bar{r}(Y \setminus \mathbb{E}^Y) = 0$.

Denote $\mathcal{G} := \hat{Y} \setminus \mathbb{E}^{\hat{Y}}$.

Theorem

$\mathcal{G}$ is a forest and $\bar{r}(\Gamma \backslash \mathcal{G}) = 0$.

Sketch of graph-theoretic proof.

- Suppose $\mathcal{G}$ has a loop, then the maximal edge in that loop is order essential. Contradiction.
- Suppose $\bar{r}(\Gamma \backslash \mathcal{G}) \neq 0$, then some component of $\Gamma \backslash \mathcal{G}$ is not a tree and not a flower. Then its fundamental group has rank at least two. With some work, this allows constructing a line in $\mathcal{G}$ that has a maximal edge, then this edge is order-essential. Contradiction.

The analytic version of (2) is easier to prove, and it works for any dimension. Use Murray-von Neumann dimension and the definition of order-inesential edges (or cells).

There are generalizations of SHNC to complexes.

Let $\hat{Y}$ be a *complex* $\hat{Y}$ with a free cocompact Γ-action. Denote

$$a_i^{(2)}(\hat{Y},\Gamma) := \dim_\Gamma \mathrm{Ker}\left(\partial : \ell^2(\Sigma_i^{\hat{Y}}) \to \ell^2(\Sigma_{i-1}^{\hat{Y}})\right),$$

$$b_i^{(2)}(\hat{Y},\Gamma) := \dim_\Gamma H_i^{(2)}(\hat{Y}) \qquad \ell^2\text{-Betti number.}$$

Here $\dim_\Gamma$ is the Murray-von Neumann dimension.

The submultiplicativity question.

(a) Under what conditions

$$a_i^{(2)}(\hat{Y} \hat{\Box} \hat{Z}, \Gamma) \leq a_i^{(2)}(\hat{Y}, \Gamma) \cdot a_i^{(2)}(\hat{Z}, \Gamma) \ ?$$

(b) Under what conditions

$$b_i^{(2)}(\hat{Y} \hat{\Box} \hat{Z}, \Gamma) \leq b_i^{(2)}(\hat{Y}, \Gamma) \cdot b_i^{(2)}(\hat{Z}, \Gamma) \ ?$$

The deep-fall question/problem:

Find many examples of free actions by a left-orderable group Γ on a complex $\hat{Y}$ satisfying the deep-fall property:

If an i-cell σ is order-essential, i.e. if $\partial\sigma \in \overline{\partial(\ell^2[\Sigma_i^{\hat{Y}} < \sigma])}$, then $\partial\sigma \in \overline{\partial(\ell^2[\mathbb{I}_i^{\hat{Y}} < \sigma])}$.

For such examples the (integral) Atiyah Conjecture holds for ∂_i.


Q.E.D.