Leafages: how to prove the Hanna Neumann Conjecture.
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Although the summer sunlight gild
Cloudy leafage of the sky,
Or wintry moonlight sink the field
In storm-scattered intricacy,
I cannot look thereon,
Responsibility so weighs me down.

William Butler Yeats, “Vacillation”
Although the summer sunlight gild
Cloudy leafage of the sky,
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William Butler Yeats, “Vacillation”.
Imagine you are in a *forest* ... Or in a *garden* ... Look around ...
You see trees and flowers around you ... Look up again ...
You see beautiful *leafage* above ...
Look closer ... You can now see leaves ... And their veins ...
We are now ready to start ...
Theorem (Hanna Neumann 1956-1957)
Suppose $\Gamma$ is a free group and $A$ and $B$ are its nontrivial finitely generated subgroups. Then

$$\text{rk} (A \cap B) - 1 \leq 2 (\text{rk} A - 1)(\text{rk} B - 1).$$

Conjecture (HNC, Hanna Neumann 1956-1957)
Suppose $\Gamma$ is a free group and $A$ and $B$ are its nontrivial finitely generated subgroups. Then

$$\text{rk} (A \cap B) - 1 \leq (\text{rk} A - 1)(\text{rk} B - 1).$$
The \textit{reduced rank} of a free group, due to Walter Neumann:

\[ \bar{r}(A) := \max \{ 0, \rk A - 1 \} \geq 0. \]

\textbf{HNC restated}

Suppose $\Gamma$ is a free group and $A$ and $B$ are its finitely generated subgroups, \textit{trivial or not}. Then $\bar{r}(A \cap B) \leq \bar{r}(A) \cdot \bar{r}(B)$.

\textbf{Conjecture (SHNC, Walter Neumann 1989-1990)}

Suppose $\Gamma$ is a free group and $A$ and $B$ are its finitely generated subgroups. Then

\[ \sum_{z \in s(A \setminus \Gamma/B)} \bar{r}(A^z \cap B) \leq \bar{r}(A) \cdot \bar{r}(B). \]
This talk is based on the following papers:

(1) *The topology and analysis of the Hanna Neumann Conjecture.* The Journal of Topology and Analysis (JTA). Proving HNC was not the goal of this paper. It rather uses analysis to **generalize the statement** SHNC, and gives **approaches to those generalizations**.

(2) *Submultiplicativity and the Hanna Neumann Conjecture.* May 2011. The proof of SHNC and of a more general result is given in the analytic language, it uses Hilbert modules and some graph theory.

(3) *Groups, graphs, and the Hanna Neumann Conjecture.* May 2011. The proof of the original SHNC purely in terms of groups and graphs.
Joel Friedman has also recently announced a proof of SHNC. Preprints:


(4) “Sheaves on Graphs and Their Homological Invariants”, April 2011.

Stallings’ diagram:
fiber product of graphs
\[ \pi_1(X) = \Gamma, \quad \pi_1(Y) = A, \quad \pi_1(Z) = B. \]

The \textit{reduced rank} of a finite graph \( Y \):

\[ \bar{r}(Y) := \sum_{K \in \text{Comp}(Y)} \max\{0, -\chi(K)\}. \]

If \( Y \) is nonempty,

\[ \bar{r}(Y) = \sum_{K \in \text{Comp}(Y)} \bar{r}(\pi_1(K)). \]

A restatement of SHNC: \( \bar{r}(S) \leq \bar{r}(Y) \cdot \bar{r}(Z) \).

To prove SHNC, we want to describe \( \bar{r}(S), \bar{r}(Y), \bar{r}(Z) \).
A tree
\[ \bar{r} = 0 \]

A flower
\[ \bar{r} = 0 \]

A forest
\[ \bar{r} = 0 \]

A garden
\[ \bar{r} = 0 \]
A graph $Y$.

An essential edge in $Y$.

An essential set in $Y$.

A maximal essential set in $Y$. 
Formally:

- An **essential edge** in $Y$: removing it decreases reduced rank exactly by 1 (rather than by 0). **Inessential** otherwise.

- An **essential set** of edges, $E \subseteq E^Y$: one that decreases reduced rank exactly by $#E$, i.e. $\bar{r}(Y \setminus E) = \bar{r}(Y) - #E$.

- A **maximal essential set**, $E \subseteq E^Y$, realizes $\bar{r}(Y)$, i.e. $\bar{r}(Y \setminus E) = \bar{r}(Y) - #E = 0$.

In other words, for a maximal essential set $E$, $\bar{r}(Y) = #E$. This gives a description of reduced rank.
How can one try to prove HNC, and fail?

In many ways!

For example, one can pick a maximal essential set in $S$, then project it to $Y$ and to $Z$.

The problem is: those projections might not be be essential sets in $Y$ or in $Z$. (Exercise: find a counterexample.)

Our plan is to make this idea work nevertheless.
A *leafage* is a map of cell complexes $\hat{S} \to \hat{Y}$ whose restriction to each component of $\hat{S}$ is injective.
How do leafages arise?

Let $\alpha : Y \to X$ and $\beta : Z \to X$ be *immersions* of complexes, defined as maps that can be extended to covers of $X$. The main example: the Stallings’ immersions of finite graphs.

Let $S$ be their fiber-product.

Let $\hat{X}$ be a complex with a free $\Gamma$-action whose quotient is $X$, for example the universal cover of $X$. In the main example, $\Gamma$ is a free group.

Let $p_X : \hat{X} \to X$ be the quotient map.

Pull this whole diagram back by $p_X$. 
The result is a *system* of complexes:

\[ S \xrightarrow{\nu} Z \]

\[ S \xrightarrow{\hat{\nu}} \hat{Z} \]

\[ Y \xrightarrow{\hat{\alpha}} \hat{X} \]

\[ \Gamma \text{ acts freely on } \hat{X}, \hat{Y}, \hat{Z}, \hat{S}, \]

with quotients \( X, Y, Z, S \).

If \( \hat{X} \) is simply connected (connected or not), then \( \hat{\alpha} \) and \( \hat{\beta} \) are leafages. Hence \( \hat{\mu} \) and \( \hat{\nu} \) are leafages as well.
Now assume $\Gamma$ is left-orderable. For example, any free group is left-orderable.

Define a $\Gamma$-invariant total order $\leq$ on $E^{\hat{X}}$:
- Pick a $\Gamma$-transversal subset $\bar{E}^{\hat{X}}$ of $E^{\hat{X}}$. Then $E^{\hat{X}} \cong \Gamma \times \bar{E}^{\hat{X}}$.
- Put either lexicographic order on $E^{\hat{X}}$.

Put pull-back orders on $E^{\hat{Y}}$, $E^{\hat{Z}}$, $E^{\hat{S}}$. These are also $\Gamma$-invariant.

The restrictions of each leafage to each component is strictly order-preserving (on edges).
Now we define *veins* in a graph \( \hat{Y} \).
Let $\sigma \in E^{\hat{Y}}$, $E \subseteq E^{\hat{Y}} \setminus \{\sigma\}$.

← A finite vein at $\sigma$ in $E$.

← An infinite vein at $\sigma$ in $E$.

If $\hat{Y}$ is a forest, all veins are infinite.

We say that $\sigma$ falls into $E$ if there is a vein at $\sigma$ in $E$. 
The most important definition:
An edge $\sigma$ in $\hat{Y}$ is \textit{order-essential} if it falls into

$$[E^{\hat{Y}} < \sigma] := \{\tau \in E^{\hat{Y}} \mid \tau < \sigma\}.$$  

[An analytic comment: this is equivalent to $\partial \sigma \in \partial(\ell^2[E^{\hat{Y}} < \sigma])].$
\textit{Order-inessential} otherwise.

$E^{\hat{Y}} := \text{the set of order-essential edges in } \hat{Y}, \quad E^Y := \Gamma \setminus E^{\hat{Y}}.$
$I^{\hat{Y}} := \text{the set of order-inessential edges in } \hat{Y}.$

$E^{\hat{Y}} \text{ and } I^{\hat{Y}} \text{ are } \Gamma\text{-invariant}.$

The most important observation:
leafages map order-essential edges to order-essential edges.
Hence \( \#E^S \leq \#E^Y \cdot \#E^Z \).

(Regardless of whether the graphs \( \hat{Y}, \hat{Z}, \hat{S} \) are forests or not.)

It remains to show that \( \#E^Y = \bar{r}(Y) \).
To show that $\#\mathbb{E}^Y = \bar{r}(Y)$, it suffices to show that $\#\mathbb{E}^Y$ is a maximal essential set in $Y$. 
(1) To show that $\#E^Y$ is essential we need:

The deep-fall property for graphs

If $\sigma \in E^{\hat{Y}}$ (i.e. if $\sigma$ falls into $[E^{\hat{Y}} < \sigma]$), then $\sigma$ falls into $[I^{\hat{Y}} < \sigma]$.

Sketch of proof:

▶ Enumerate $[E^{\hat{Y}} < \sigma]$ as $\{\sigma_1, \sigma_2, \ldots\}$.
▶ For each $n$, relabel $\{\sigma_1, \ldots, \sigma_n\}$ as $\{\tau_1, \ldots, \tau_n\}$ so that $\tau_1 > \ldots > \tau_n$.
▶ $\sigma$ has a vein in $[E^{\hat{Y}} < \sigma]$.
▶ Each $\tau_i < \sigma$ and has a vein in $[E^{\hat{Y}} < \tau_i]$.
▶ Replace $\tau_i$’s with their veins one by one to construct a vein at $\sigma$ that misses $\{\tau_1, \ldots, \tau_n\} = \{\sigma_1, \ldots, \sigma_n\}$.
▶ Then there is a limiting vein at sigma that misses $[E^{\hat{Y}} < \sigma]$.

Then remove the edges of $\#E^Y$ from $Y$ one by one and check that reduced rank decreases exactly by 1 at each step. (Use the fact that $\hat{Y}$ is a forest.)
(2) To show that $\#E^Y$ is maximal we need to show that $ar{r}(Y \setminus E^Y) = 0$.

Denote $\mathcal{G} := \hat{Y} \setminus E^{\hat{Y}}$.

**Theorem**

$\mathcal{G}$ is a forest and $\bar{r}(\Gamma \setminus \mathcal{G}) = 0$.

**Sketch of graph-theoretic proof.**

- Suppose $\mathcal{G}$ has a loop, then the maximal edge in that loop is order essential. Contradiction.

- Suppose $\bar{r}(\Gamma \setminus \mathcal{G}) \neq 0$, then some component of $\Gamma \setminus \mathcal{G}$ is not a tree and not a flower. Then its fundamental group has rank at least two. With some work, this allows constructing a line in $\mathcal{G}$ that has a maximal edge, then this edge is order-essential. Contradiction.
The analytic version of (2) is easier to prove, and it works for any dimension. Use Murray-von Neumann dimension and the definition of order-inesential edges (or cells).

There are generalizations of SHNC to complexes.
Let $\hat{Y}$ be a complex $\hat{Y}$ with a free cocompact $\Gamma$-action. Denote

$$a_i^{(2)}(\hat{Y}, \Gamma) := \dim_\Gamma \ker \left( \partial : \ell^2(\Sigma_i^{\hat{Y}}) \to \ell^2(\Sigma_{i-1}^{\hat{Y}}) \right),$$

$$b_i^{(2)}(\hat{Y}, \Gamma) := \dim_\Gamma H_i^{(2)}(\hat{Y}) \quad \text{\(\ell^2\)-Betti number.}$$

Here $\dim_\Gamma$ is the Murray-von Neumann dimension.
The submultiplicativity question.

(a) Under what conditions
\[ a_i^{(2)}(\hat{Y} \sqcap \hat{Z}, \Gamma) \leq a_i^{(2)}(\hat{Y}, \Gamma) \cdot a_i^{(2)}(\hat{Z}, \Gamma) \] ?

(b) Under what conditions
\[ b_i^{(2)}(\hat{Y} \sqcap \hat{Z}, \Gamma) \leq b_i^{(2)}(\hat{Y}, \Gamma) \cdot b_i^{(2)}(\hat{Z}, \Gamma) \] ?

The deep-fall question/problem:
Find many examples of free actions by a left-orderable group \( \Gamma \) on a complex \( \hat{Y} \) satisfying the deep-fall property:
If an \( i \)-cell \( \sigma \) is order-essential, i.e. if \( \partial \sigma \in \partial(\ell^2[\Sigma \hat{Y} < \sigma]) \), then \( \partial \sigma \in \partial(\ell^2[I_i \hat{Y} < \sigma]) \).

For such examples the (integral) Atiyah Conjecture holds for \( \partial_i \).