Algebraic & definable closure in free groups

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In particular, $acl(A)$ is finitely generated, quasiconvex and hyperbolic.

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(1) *Let Λ be the abelian JSJ-decomposition of K relative to $acl(A)$. Then the vertex group containing $acl(A)$ in Λ is exactely $acl(A)$.*

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Let Γ be a torsion-free hyperbolic group and A a nonabelian subgroup of Γ. Let Γ^* be a nonprincipal ultrapower of Γ. Let $K \leq \Gamma^*$ be a finitely generated subgroup such that $acl(A) \leq K$. Then K can be constructed from $acl(A)$ by a finite sequence of amalgamated free products and HNN-extensions along abelian subgroups.

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For $m \in \mathbb{N}$, we set

$$(*) \qquad \varphi_m(\bar{x}) := W(\bar{x}) \wedge \bigwedge_{0 \leq i \leq m} v_i(\bar{x}) \neq 1.$$

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Remark. $Gal(G/A)$ is a subgroup which contains $acl(A)$.

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- $G = \langle G, t | B^t = C \rangle$. Similar. □

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Proof Since $|Aut(\Gamma/A) : Mod(G/A)| < \infty$, there exist automorphisms $f_1, \ldots, f_l$ of Γ such that for any $f \in Aut(F/A)$, there exists a modular automorphism $\sigma \in Mod(\Gamma/A)$ such that $f = f_i \circ \sigma$ for some i.

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Let Γ be a torison-free hyperbolic group and A a nonabelian subgroup such that Γ is freely A-indecomposable. Then $Gal(G/A)$ coincides with the vertex group containing A in the cyclic JSJ-decomposition of Γ (relative to A).

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Since h is a notrivial automorphism of K of finite order, K is freely $dcl(A)$-decomposable; a contradiction.

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where D is a free group. Since $dcl(A) \leq K \cap h(K)$, it follows that for some i, $g_i = 1$. Since K is $dcl(A)$-freely indecompsable, we find that $h(K) = h(K) \cap K$ and thus $h(K) \leq K$. In particular $h(a) \in K$.
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Since h is a notrivial automorphism of K of finite order, K is freely $dcl(A)$-decomposable; a contradiction. Hence in each case, we get a contradiction. Therefore $dcl(A) = K$ as required. □

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Then $F \models \varphi(y)$. Let $\gamma \in F$ such that $F \models \varphi(\gamma)$. Then the map defined by $f(y) = \gamma$, $f(t) = \alpha$ and the identity on A extends to a homomorphism of F and thus, by the first propertie, $\gamma = y^{\pm 1}$.

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$$g(v) = ay^{-1}by^{-1}ayby = ay^{-1}by^{-1}aybyay^{-1}by^{-1}(ay^{-1}by^{-1})^{-1}$$

$$= dvd^{-1},$$

where $d = ay^{-1}by^{-1}$.

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we get $g \in Aut(F/A)$ with $g(y) = y^{-1}$. Now if $\gamma \in H \setminus A$ then y appears in the normal form of γ and thus $g(\gamma) \neq \gamma$ as required. □

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(3) Any nontrivial automorphism $f \in Aut(F/A)$ satisfies $f(y) = y^{-1}$.