Golod-Shafarevich groups

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Outline

1. Golod-Shafarevich groups
   - Motivation: class field tower problem
   - Golod-Shafarevich algebras
   - Golod-Shafarevich groups
   - Structure of Golod-Shafarevich groups
   - Further applications of Golod-Shafarevich groups
   - Generalized Golod-Shafarevich groups

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$$K = K^{(0)} \subseteq K^{(1)} \subseteq K^{(2)} \subseteq \dots$$

is defined by $K^{(i)} = \mathbb{H}(K^{(i-1)})$.

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Problem (Class field tower (CFT) problem)

Does there exist K with infinite class field tower?

p-class field towers

- Fix a prime p.
- Let $\mathbb{H}_p(K)$ be the p**-class field** of K= maximal unramified Galois extension of K such that $Gal(\mathbb{H}_p(K)/K)$ is an elementary abelian p-group.
- Let $\widehat{K}_p = \cup K_p^{(i)}$. Then $\widehat{K}_p$ is the max. unramified p-extension of K.
- Let $G_{K,p} = Gal(\widehat{K}_p/K)$.
- Thus, to solve CFT it suffices to find K with $G_{K,p}$ **infinite**.

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Class field tower problem

In his 1963 IHES paper, Shafarevich studied presentations for $G_{K,p}$ and proved that $r(G_{k,p}) \leqslant d(G_{k,p}) + \rho(K)$ where

- $d(G_{K,p})$ is the minimal number of generators of $G_{K,p}$
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This result implies that

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For every p there exists a sequence $\{K_n\}$ of number fields such that

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If G is a finite p-group, then $r(G) \geqslant (d(G) - 1)^2/4$.
Therefore, CFT problem has positive solution.

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- $K\langle U\rangle = K\langle u_1, \ldots, u_d\rangle = \oplus_{n=0}^{\infty} K\langle U\rangle_n$.
- R a subset of $K\langle U\rangle$ consisting of homogeneous elements of positive degree
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Then

$$I = \oplus_{n=0}^{\infty} I_n \quad \text{where} \quad I_n = I \cap K\langle U\rangle_n$$
$$A = \oplus_{n=0}^{\infty} A_n \quad \text{where} \quad A_n = K\langle U\rangle_n / I_n.$$

Golod-Shafarevich algebras: graded case

- Let $r_n = |\{r \in R : deg(r) = n\}|$ and $a_n = \dim_K A_n$.
- $H_R(t) = \sum_{n=1}^{\infty} r_n t^n$ and $H_A(t) = \sum_{n=0}^{\infty} a_n t^n$

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Definition

- The condition (***) is called the **GS condition**.
- A graded algebra B is called a **GS algebra** if it has a presentation satisfying the GS condition.

Finite-dimensional graded algebras require many relations

- Let B be a graded algebra over K with $\dim_K B < \infty$.
- Let $\langle U|R\rangle$ be a minimal presentation of B. Then $r_1 = 0$ (no degree 1 relators).
- Hence for any $\tau \in (0,1)$ we have

$$0 \leqslant 1 - d\tau + H_R(\tau) \leqslant 1 - d\tau + |R|\tau^2.$$

- Therefore, $|R| \geqslant d^2/4$.

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- Let $K\langle\langle U\rangle\rangle = K\langle\langle u_1,\ldots,u_d\rangle\rangle$ (power series)
- For $f \in K\langle\langle U\rangle\rangle$ let $deg(f)$ be the smallest integer n s.t. f involves a monomial of degree n. Let $K\langle\langle U\rangle\rangle_n = \{f \in K\langle\langle U\rangle\rangle : deg(f) \geqslant n\}$.
- Let R be a subset of $K\langle\langle U\rangle\rangle$ consisting of elements of pos. degree
- I the closure of the ideal of $K\langle\langle U\rangle\rangle$ generated by R.
- $A = K\langle\langle U\rangle\rangle/I$
- $r_n = |\{r \in R : deg(r) = n\}|$,
- $A_n = \pi(K\langle\langle U\rangle\rangle_n/K\langle\langle U\rangle\rangle_{n+1})$, where $\pi : K\langle\langle U\rangle\rangle \to A$ is the natural projection, and $a_n = \dim_K A_n$,
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Theorem (Golod-Shafarevich inequality: non-graded case)

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Zassenhaus degree function on free groups

- Fix a prime p (for the rest of the talk).
- Let $X = \{x_1, \ldots, x_d\}$ be a finite set and $F(X)$ the free group on X.
- Let $U = \{u_1, \ldots, u_d\}$ and consider the **Magnus embedding** of $F(X)$ into $\mathbb{F}_p\langle\langle U\rangle\rangle^*$ given by

$$x_i \mapsto 1 + u_i.$$

- For $f \in F(X)$ set $D(f) = deg(f - 1)$

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Basic properties of the degree function D.

- $D(f) \geqslant 1$ for any $f \in F(X)$
- $D([f,g]) \geqslant D(f) + D(g)$ for any $f, g \in F(X)$ where $[f,g] = f^{-1}g^{-1}fg$
- $D(f^p) = p \cdot D(f)$ for any $f \in F(X)$.

Golod-Shafarevich groups

Definition

A f.g. group G is a **GS group** if it has a presentation $\langle X|R\rangle$ with the following property: there exists $0 < \tau < 1$ such that

$$1 - |X|\tau + H_R(\tau) < 0$$

where $H_R(t) = \sum_{n=1}^{\infty} r_n t^n$ and $r_n = |\{r \in R : D(r) = n\}|$.

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Remark: Similarly, one defines GS pro-p groups. The only difference is that $D(f)$ has to be defined for $f \in F(X)_{\hat{p}}$, the **free pro-p group on X.** This causes no problem since $F(X)_{\hat{p}}$ can be realized as the closure of $F(X)$ inside $\mathbb{F}_p\langle\langle u_1, \ldots, u_n\rangle\rangle$.

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Theorem (Golod-Shafarevich)

Let G be a Golod-Shafarevich group. Then the pro-p completion of G is infinite. In particular, G itself is infinite.

Proof: connection with Golod-Shafarevich algebras

- If G is a f.g. group, consider its completed group algebra

$$\mathbb{F}_p[[G]] = \varprojlim \mathbb{F}_p[G/N]$$

where N runs over all normal subgroups of p-power index.

- Assume now that G is a GS group. Then one can show that $\mathbb{F}_p[[G]]$ is a GS algebra (in the non-graded sense).
- By GS inequality $\mathbb{F}_p[[G]]$ is infinite, so G is infinite. Moreover, G must have infinitely many normal subgroups of p-power index, so its pro-p completion is infinite.
- The same argument shows that GS pro-p groups are infinite.
- One can say much more: GS groups are "big" in many different ways.

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Infinite finitely generated torsion groups

Problem (General Burnside Problem, 1902)

Let G be a f.g. torsion group. Does G have to be finite?

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- Let $F = F(x_1, x_2)$. Enumerate all elements of F: $f_1, f_2, f_3, \ldots$.
- Let $G = \langle X | R \rangle$ where $X = \{x_1, x_2\}$ and $R = \{f_1^{p^{n_1}}, f_2^{p^{n_2}}, \ldots\}$ for some integers $n_1, n_2, \ldots$. By construction G is torsion.
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- Note that $1 - |X|\tau + H_R(\tau) \leqslant 1 - 2\tau + \sum_{i=1}^{\infty} \tau^{p^{n_i}} < 0$ whenever $1/2 < \tau < 1$ and $\{n_i\}$ are large enough, so we can make G a GS group.

Outline

1. Golod-Shafarevich groups
   - Motivation: class field tower problem
   - Golod-Shafarevich algebras
   - Golod-Shafarevich groups
   - Structure of Golod-Shafarevich groups
   - Further applications of Golod-Shafarevich groups
   - Generalized Golod-Shafarevich groups

Quotients of Golod-Shafarevich groups

If G is a GS group and (P) is some group-theoretic property, one can often construct a quotient of G which has (P) and is also GS (in particular, it is infinite).

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Theorem (Wilson, 1991)

Every GS group has a torsion quotient which is also GS.

Theorem (E-Jaikin, 2012)

Every GS group has a LERF quotient which is also GS.

Theorem (Myasnikov-Osin, 2012)

Every recursively presented GS group has a quotient which is a Dehn monster and is also GS.

Growth and subgroups of Golod-Shafarevich groups

Proposition (Bartholdi-Grigorchuk, 2000)

GS groups have uniformly exponential growth.

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Theorem (Zelmanov, 2000)

Let G be a GS pro-p group. Then G contains a non-abelian free pro-p subgroup.

Subgroup growth of Golod-Shafarevich pro-p groups

If G is a f.g. pro-p group, let $a_n(G)$ be the number of subgroups of index n in G (note that $a_n(G) = 0$ unless $n = p^k$).

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- If G is p-adic analytic, then $\log\log(a_n(G)) \sim \log\log(n)$ (polynomial growth)
- (Shalev, 1992) If G is non-p-adic analytic, then $\log\log(a_n(G)) \geqslant (2-\varepsilon)\log\log(n)$ for infinitely many n.
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Problem

Does there exist a GS group G

(a) *with subexponential subgroup growth?*

(b) *s.t.* $\log\log(a_n(G)) \sim \log(n)^\alpha$ *where* $\alpha < 1$*?*

Outline

Golod-Shafarevich groups and Kazhdan's property (T)

Theorem (E)

- *A (2008) There exist GS groups with Kazhdan's property* (T).
- *B (2011) Every GS group has an infinite quotient with property* (T). *In particular, GS groups are never amenable.*

- Part B "confirms" the general philosophy that GS groups should be "big".
- For the same reason Part A is somewhat surprsing since groups with property (T) should not be "big".
- Despite this contrast, the proof of Part B uses Part A.
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Golod-Shafarevich groups in 3-dimensional topology.

Let M be a compact orientable 3-manifold and $G = \pi_1(M)$. Then G has a presentation $\langle X|R\rangle$ with $|X| = |R|$.

This condition implies that G is GS provided that $d(G_{\hat{p}}) \geqslant 5$.

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Lubotzky (1983) used this result to prove that fundamental groups of compact orientable hyperbolic 3-manifolds (which are just cocompact torsion-free lattices in $SL_2(\mathbb{C})$) do not have CSP. This was a major open problem known as Serre conjecture.

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Conjecture (Lubotzky-Sarnak, late 1980's)

If M is hyperbolic, then $\pi_1(M)$ does not have property (τ).

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Positive application of Theorem A

Corollary (E, 2008)

There exist f.g. **residually finite** *torsion non-amenable groups.*

Proof.

- Let G be a GS group with property (T).
- G has a torsion quotient G' which is still GS. G' also has (T) being a quotient of G.
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Other examples of residually finite torsion non-amenable groups were constructed by Osin (2011) and Puchta (2011).

Constructing residually finite "almost Tarski monsters"

Theorem (Ol'shanskii, 1980)

For every sufficiently large prime p there is an infinite group G in which every **proper** *subgroup has order p.*

Such groups are called **Tarski monsters**.

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Theorem (E-Jaikin, 2012)

Every GS group has a quotient G s.t.

- *G is an infinite residually finite torsion group*
- *every* **finitely generated** *subgroup of G is either finite or of finite index.*

Using Golod-Shafarevich groups to construct "exotic examples"

Let (P) and (Q) be group-theoretic properties. Suppose that

(i) (Q) is preserved by quotients

(ii) There exists a GS group with (Q)

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Example (EJ)

There exists an infinite group which is LERF and has property (T).

Here (P) =LERF and $(Q) = (T)$.

Outline

- The definition of GS groups can be restated as follows: G is GS if there exists a presentation $\langle X|R\rangle$ of G and $\tau \in (0,1)$ s.t.

$$1 - W(X) + W(R) < 0$$

where $W(S) = \sum_{s\in S} \tau^{D(s)}$ for any $S \subseteq F(X)$.

- Recall that $D(s) = \deg(s-1)$ where $F(X)$ sits inside $\mathbb{F}_p\langle\langle u_1,\ldots,u_n\rangle\rangle$ via the Magnus embedding.
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Generalized Golod-Shafarevich groups

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A group G is called a **generalized GS group (GGS group)** if it has a presentation $\langle X|R\rangle$ such that $1 - W(X) + W(R) < 0$ for some weight function W.

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Thus, generalized GS groups = group of positive weighted deficiency, and they also generalize groups of deficiency > 1.

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Theorem (Baumslag-Pride 1978)

If G is a group of deficiency > 1, then G is **large***, that is, G has a finite index subgroup which homomorphically maps onto a non-abelian free group.*

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Problem

Find a counterpart of the Baumslag-Pride Theorem for generalized GS groups (considered as groups of positive weighted deficiency).