# Homogeneity of the free group

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Problems about (free and hyperbolic) groups coming from first order logic.

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First-order logic on a group $G$: studying **first-order formulas** on $G$, which should be thought of as “generalized equations“.

# Plan of the talk

- First-order formulas
- Background: Tarski problem
- Homogeneity
- Homogeneity of $\mathbb{F}_k$: some idea of the proof
- A small detour: elementary embeddings
- Non-homogeneity of surface groups.

# First-order formulas

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## Examples

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The simplest example of a first order formula on groups is an equation. But we also allow:

- inequations;
- conjunction and disjunction of equations and inequations;
- using quantifiers on the variables.

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$$\forall y \quad xyx^{-1}y^{-1} = 1 \text{ and } x \neq 1$$
$$\exists z \quad z^2y^{-1} \neq 1 \text{ or } z^3 = 1$$

# First-order formulas

**Important:** the variables $x, y, \dots$ always represent elements of the group. They cannot represent integers, or subsets of the group.

## Examples

The following are NOT first-order formulas:

- $\forall x \, \exists n \; x^n = 1$;
- $\exists n \, \exists x_1 \exists y_1 \dots \exists x_n \exists y_n \;\; z = [x_1, y_1] \dots [x_n, y_n]$;
- $\forall H \leq G \; (\forall x \; xHx^{-1} = H) \Rightarrow (H = 1 \text{ or } H = G)$.

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## Definition

A variable $z$ that appears in a formula $\phi$ is said to be free in $\phi$ if neither $\forall z$ nor $\exists z$ appear before it.
If a first-order formula $\phi$ has free variables $x_1, \ldots, x_n$, we will denote it $\phi(x_1, \ldots, x_n)$.

# First-order formulas

A first order formula without free variables is also called a **sentence**.

**Definition**

Given a group $G$ and a sentence $\phi$, we say $G$ satisfies $\phi$ if $\phi$ is true on $G$. We then write $G \models \phi$.

**Exemple:** $\phi : \ \forall x \, \forall y \ xyx^{-1}y^{-1} = 1.$

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$G$ a group. Some properties of $G$ can be expressed by first-order sentences (e.g. abelianity), some others cannot.
**Question:** How much can we say about a group just with first-order sentences?

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- Elementary embeddings
- Non-homogeneity of surface groups.

# Tarski problem

**Definition**

The **first-order theory** of a group $G$ is the set $\mathrm{Th}(G)$ of sentences satisfied by $G$.

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- If $G_1$ finitely generated abelian and $G_2$ finitely generated then $G_1 \simeq G_2$.

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**Question:** If $G_1 = \mathbb{F}_k$ the free group of rank $k$, and $G_2$ finitely generated? Is $G_2$ free as well? If it is free, does it have the same rank?

**Tarski problem (1945):** Do free groups of different rank have the same first-order theory?

# Tarski problem

**Theorem (Kharlampovich-Myasnikov, Sela)**

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**Theorem (Sela)**

Let $\Gamma$ be a torsion free hyperbolic group. Let $G$ be a finitely generated group. If $\mathrm{Th}(G) = \mathrm{Th}(\Gamma)$, then $G$ is torsion free hyperbolic.

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We fix an element $g$ in a group $G$. We are interested in the properties of $g$ that can be expressed by a first-order formula.

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$$\phi_1(g) \ : \ \forall y \ \ gyg^{-1}y^{-1} = 1$$

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Note that if $\sigma$ is an automorphism of $G$, then $\sigma(g)$ and $g$ have the same type. Conversely?

# Homogeneity

**Theorem (Pillay)**

Let $\mathbb{F}_k$ be the free group on $a_1, \ldots, a_k$. If an element $u$ of $\mathbb{F}_k$ has the same type as $a_1$, then $u$ is primitive, in particular there is an automorphism $\sigma$ of $\mathbb{F}_k$ with $\sigma(u) = a_1$.

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The **type** $\mathrm{tp}^G(g_1, \ldots, g_l)$ of $(g_1, \ldots, g_l)$ in $G$ is the set of first-order formulas with $l$ free variables $\phi(x_1, \ldots, x_l)$ such that $G$ satisfies $\phi(g_1, \ldots, g_l)$.

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**Definition**

A countable group $G$ is **homogeneous** if for all $l \in \mathbb{N}$,

$$\mathrm{tp}^G(g_1, \ldots, g_l) = \mathrm{tp}^G(g'_1, \ldots, g'_l)$$

$\iff$ there is $\sigma \in \mathrm{Aut}(G)$ such that $\sigma(g_i) = g'_i$ for $1 \leq i \leq l$.

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**Theorem (P.-Sklinos)**

The fundamental group $\pi_1(\Sigma)$ of a surface $\Sigma$ of characteristic at most $-3$ is not homogeneous.

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Easy to find a homomorphism $\theta : \mathbb{F}_k \to \mathbb{F}_k$ with $\theta(u) = v$.

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$$\begin{aligned} \theta(u) &= \theta(w_u(a_1, \dots, a_k)) \\ &= w_u(\theta(a_1), \dots, \theta(a_k)) \\ &= w_u(b_1, \dots, b_k) = v. \end{aligned}$$

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Enough to find an **injective** homomorphism $\theta : \mathbb{F}_k \to \mathbb{F}_k$ with $\theta(u) = v$.

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$\mathbb{F}_k = \langle a_1, \ldots, a_k \rangle$. Let $u, v \in \mathbb{F}_k$ such that $\mathrm{tp}^{\mathbb{F}_k}(u) = \mathrm{tp}^{\mathbb{F}_k}(v)$ and $\mathbb{F}_k$ freely indecomposable with respect to $\langle u \rangle$, and to $\langle v \rangle$.

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Take $b_1, b_2$ solution, and $\theta$ defined by $\theta(a_j) = b_j$, then $\theta(u) = v$.
$\theta(\mathbb{F}_2) = \langle b_1, b_2 \rangle$ is free of rank 2.
$\mathbb{F}_2 \overset{\theta}{\twoheadrightarrow} \theta(\mathbb{F}_2) \simeq \mathbb{F}_2$ but free groups are Hopfian so $\theta$ is injective.

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Equivalently, if we define

$$\eta : \mathbb{F}_2 \twoheadrightarrow \mathbb{F}_2 / \langle\langle [a_1, a_2] \rangle\rangle$$

then $\theta : \mathbb{F}_k \to \mathbb{F}_k$ is injective $\iff$ it does not factor through $\eta$.

In general case, we will use

**Theorem**

$u, v \in \mathbb{F}_k$ and $\langle u \rangle$ is not contained in a proper free factor of $\mathbb{F}_k$. There exists a finite set of proper quotients $\eta_j : \mathbb{F}_k \twoheadrightarrow Q_j$ such that any homomorphism $\theta : \mathbb{F}_k \to \mathbb{F}_k$ such that $\theta(u) = v$ which is not injective factors through one of the quotients $\eta_j$ **after precomposition by an element** $\sigma$ **of** $\mathrm{Aut}_{\langle u \rangle}(\mathbb{F}_k)$

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**Idea:** Use JSJ decomposition of $\mathbb{F}_k$ with respect to $\langle u \rangle$.

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**Question**

Does an element $h$ of $H$ have the same properties on $H$ and on $G$?

**Example:** $\phi(x) : \forall y \; xyx^{-1}y^{-1} = 1$.
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**Definition**

The embedding of $H$ in $G$ is **elementary** if for any $k$-uple $(h_1, \dots, h_k)$ of $H$:

$$\mathrm{tp}^G(h_1, \dots, h_k) = \mathrm{tp}^H(h_1, \dots, h_k).$$

**Remark:** $\Rightarrow \mathrm{Th}(H) = \mathrm{Th}(G)$: if $\phi$ is a sentence satisfied by $G$, $\psi(x)$ : "$\phi$ and $x = x$" is in the type of any element of $H$.

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**Theorem (Kharlampovich-Myasnikov, Sela)**

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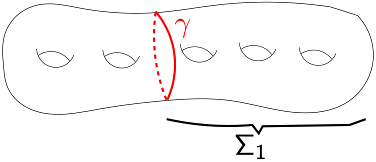
**Question**

Elementary subgroups of surface groups?

# Elementary embeddings

Let $S = \pi_1(\Sigma)$, and $S_1 = \pi_1(\Sigma_1)$.



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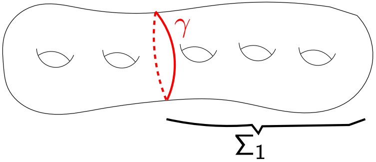


$\phi(\gamma)$ : $\exists x_1 y_1 x_2 y_2 \;\; \gamma = [x_1, y_1][x_2, y_2]$ is true on $S$, but not on $S_1$.

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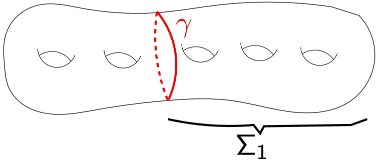


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**Theorem**

$\Sigma$ oriented hyperbolic surface. $H$ is elementary in $S = \pi_1(\Sigma)$ $\iff$ it is a free factor of $\pi_1(\Sigma_1)$ where $\Sigma_1$ is a subsurface of $\Sigma$ such that:

- $\Sigma_1^c$ is connected;
- $|\chi(\Sigma_1)| \leq |\chi(\Sigma_1^c)|$.

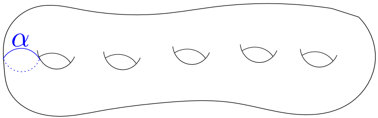
# Plan of the talk

- First-order formulas
- Background: Tarski problem
- Homogeneity
- Homogeneity of $\mathbb{F}_k$: some idea of the proof
- Elementary embeddings
- Non-homogeneity of surface groups

# Proof of non-homogeneity of surfaces

**Theorem (P.-Sklinos)**

The fundamental group $\pi_1(\Sigma)$ of a surface $\Sigma$ of characteristic at most $-3$ is not homogeneous.



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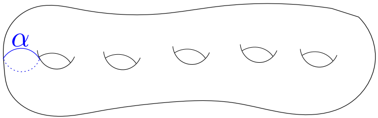


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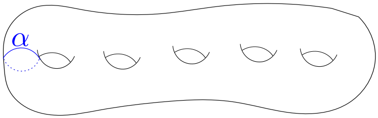


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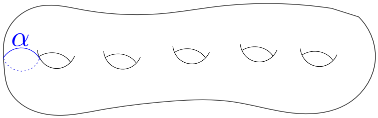


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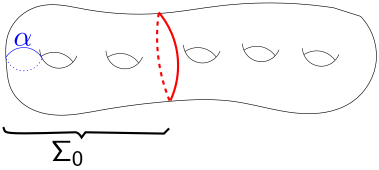


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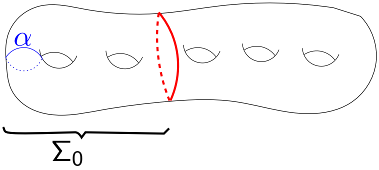


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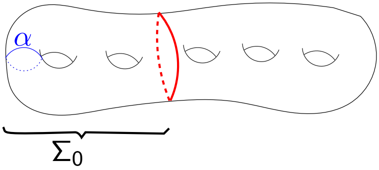

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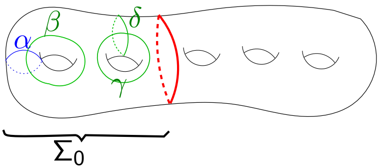


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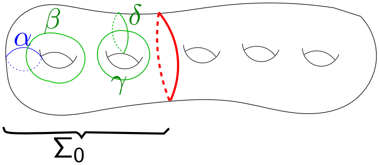


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**Claim:** $P^{cyc}$ grows exponentially (i.e: number of elements of $P^{cyc}$ of length $\leq n$ grows exponentially in $n$).

**Proof:** it contains the set $\{\alpha w(\beta, \gamma, \delta) \mid w \text{ a word in } \beta, \gamma, \delta\}$.

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