Geometric approach to the braid conjugacy problem

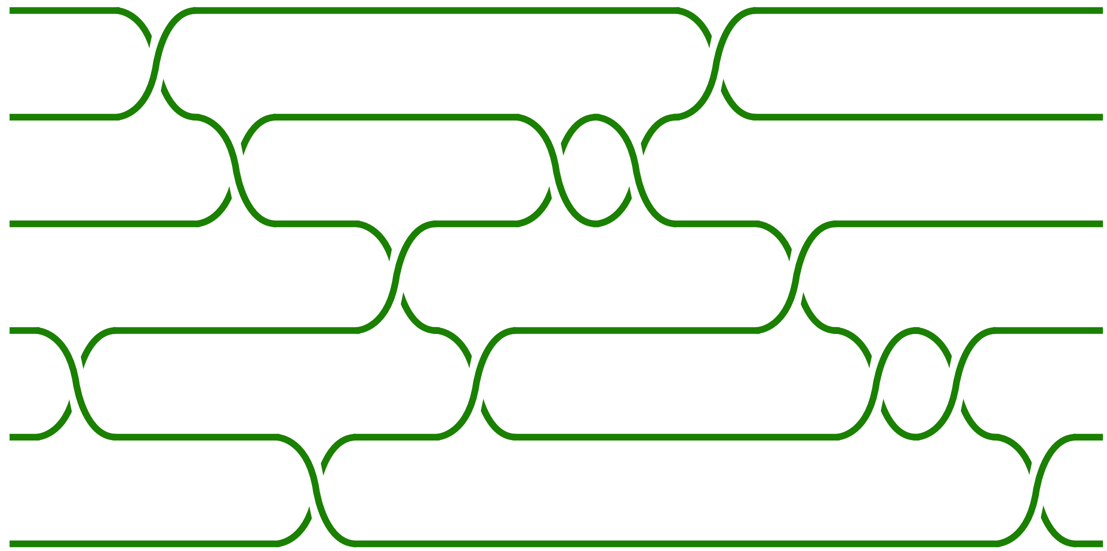
Ivan Dynnikov

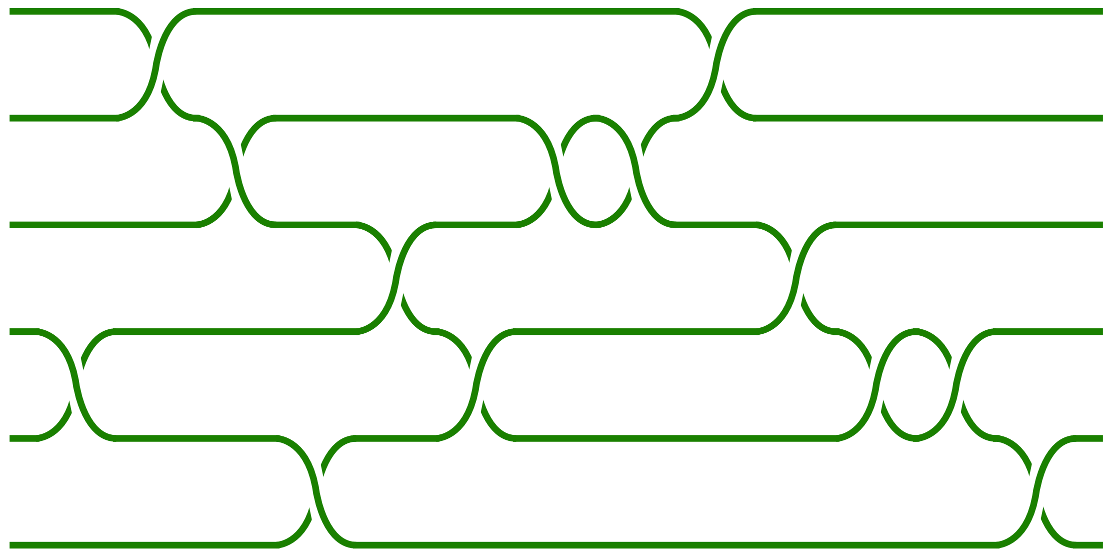
Moscow State University

Braid group B_n:

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \,\middle|\, \begin{array}{cc} \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & 1 \leqslant i \leqslant n-2 \end{array} \right\rangle$$

A braid:



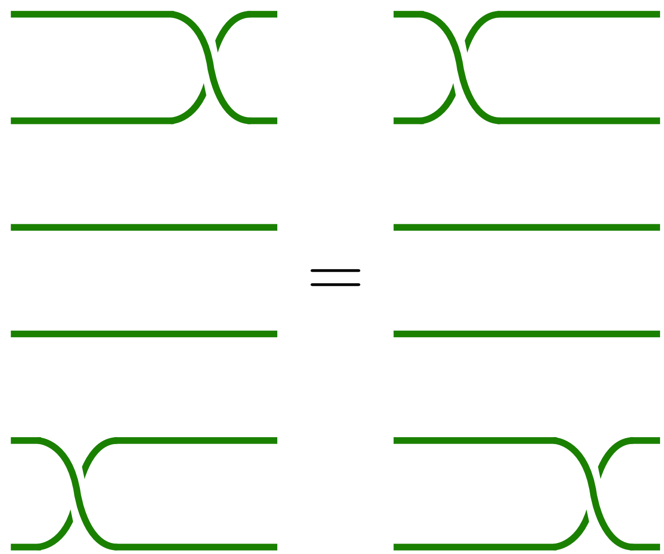


$$\sigma_2\sigma_5^{-1}\sigma_4\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_4^2\sigma_5^{-1}\sigma_3^{-1}\sigma_2^{-2}\sigma_1^{-1}$$

Relations:

$$\sigma_i \sigma_i^{-1} = 1 = \sigma_i^{-1} \sigma_i$$

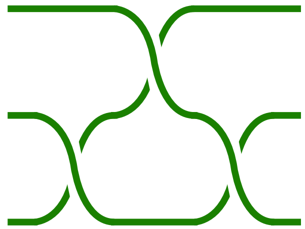
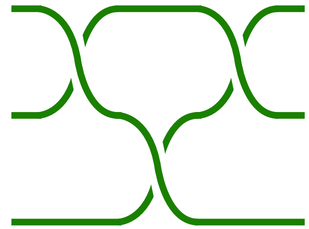
Relations:



$\sigma_i\sigma_j \quad = \quad \sigma_j\sigma_i$

$$|i-j|>1$$

Relations:

 $=$ 

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

Conjugacy Decision Problem for B_n:
given $b_1, b_2 \in B_n$ decide whether $\exists c \in B_n$ s.t. $b_1 = cb_2c^{-1}$

Conjugator Search Problem for B_n:
given $b_1, b_2 \in B_n$ that are conjugate find $c \in B_n$ s.t.
$b_1 = cb_2c^{-1}$

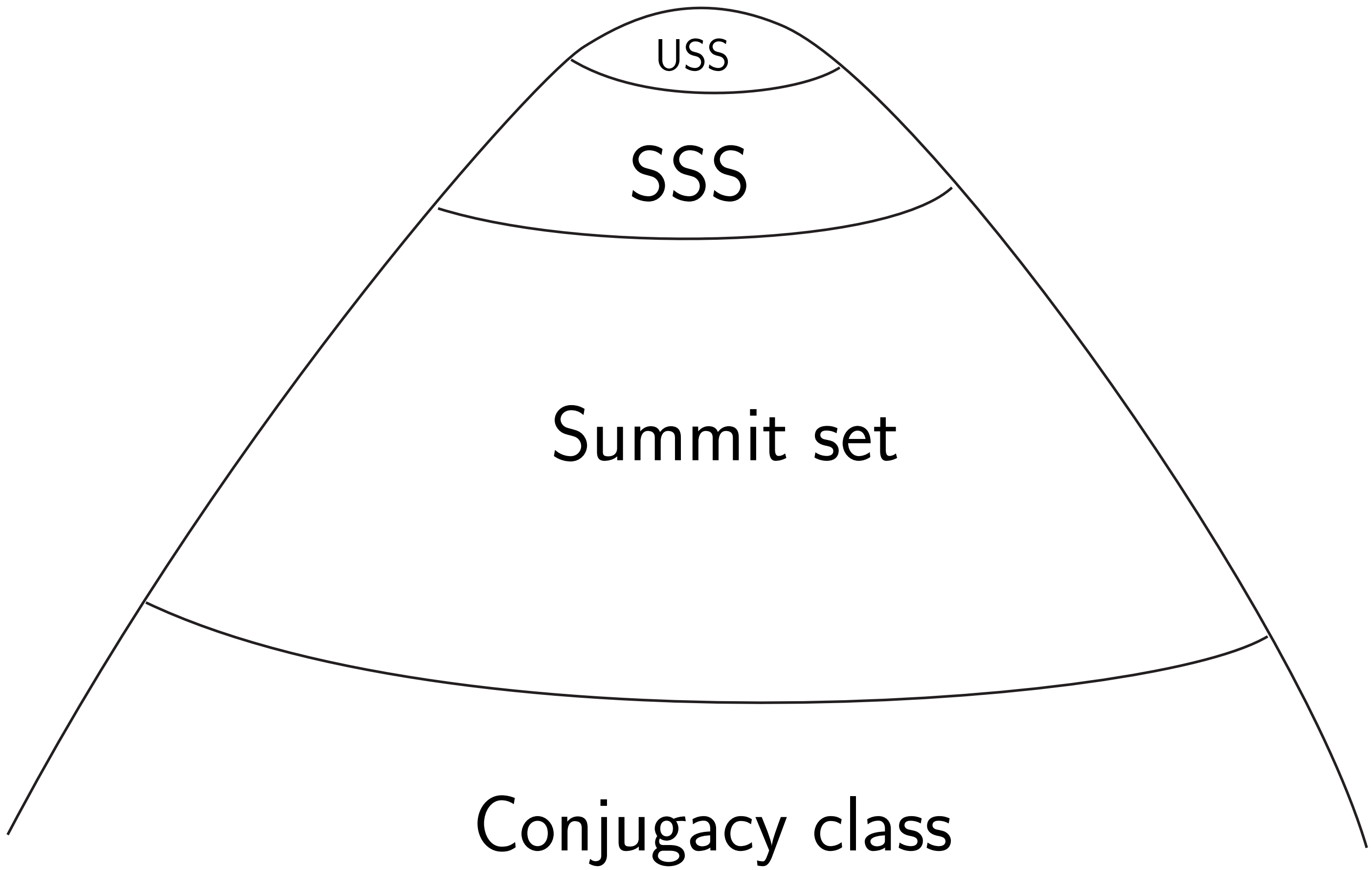
Solution:

- F.A.Garside, 1969

Given a braid b, the algorithms computes the Summit Set of the conjugacy class of b. This is a finite subset of the conjugacy class. It is (usually) exponentially large in the size of the input.

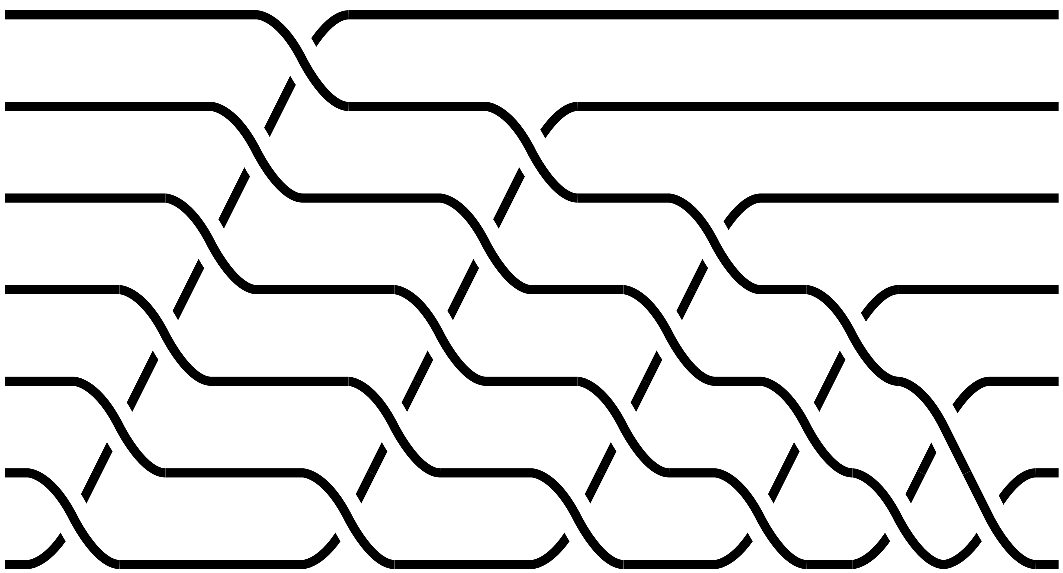
Improvements:

- W. P. Thurston, 1992 (greedy normal form)
- E. A. El-Rifai and H. R. Morton, 1994 (cycling/decycling, super summit set)
- J. Birman, K.H. Ko, and S.J. Lee, 1998 (better generating set)
- N. Franco and J. Gonzales-Meneses, 2001 (minimal simple conjugations)
- V. Gebhardt, 2003 (ultra summit set)



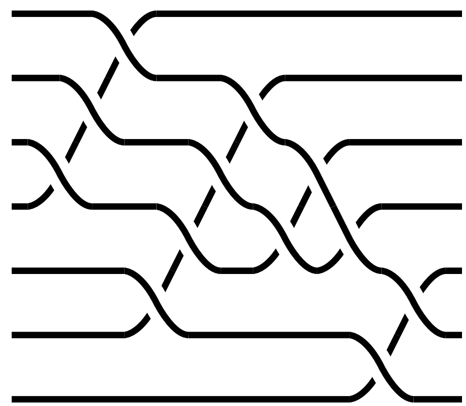
USS $\subset$ SSS $\subset$ SS $\subset$ Conjugacy class

Garside fundamental braid Δ_n:



The center of B_n is generated by Δ_n^2.

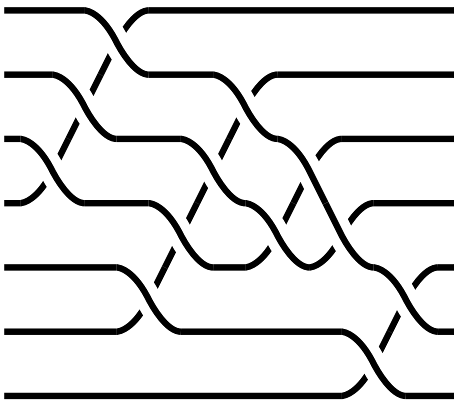
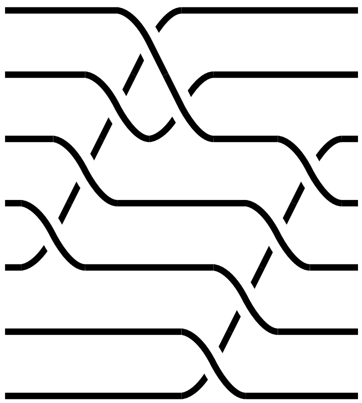
Permutation braid: only positive crossings, any two strands cross at most once.

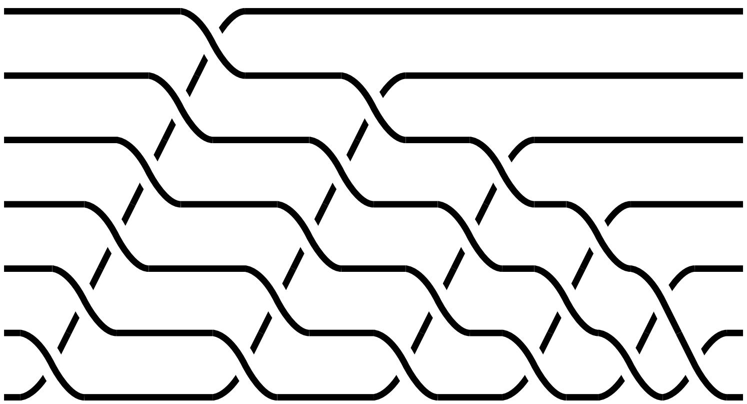


$$\{\text{permutation braids from } B_n\} \leftrightarrow S_n$$

For $\pi_1, \pi_2 \in S_n$ s.t. $|\pi_1| + |\pi_2| = |\pi_1 \circ \pi_2|$ we have

$$b_{\pi_1} b_{\pi_2} = b_{\pi_1 \pi_2}$$


$\cdot$

$=$



Thurston's left greedy form of a braid:

$$b = \Delta^k b_1 b_2 \dots b_m,$$

where:

- Δ is the Garside fundamental braid;
- b_i are permutation braids;
- $k \in \mathbb{Z}$ is maximal possible;
- each b_i is the maximal left tail of $b_i b_{i+1} \dots b_m$.

Cycling:

$$b = \Delta_n^k b_1 b_2 \ldots b_m \mapsto \Delta_n^k b_2 \ldots b_m b_1',$$

where $b_1' = \Delta_n^k b_1 \Delta_n^{-k}$.

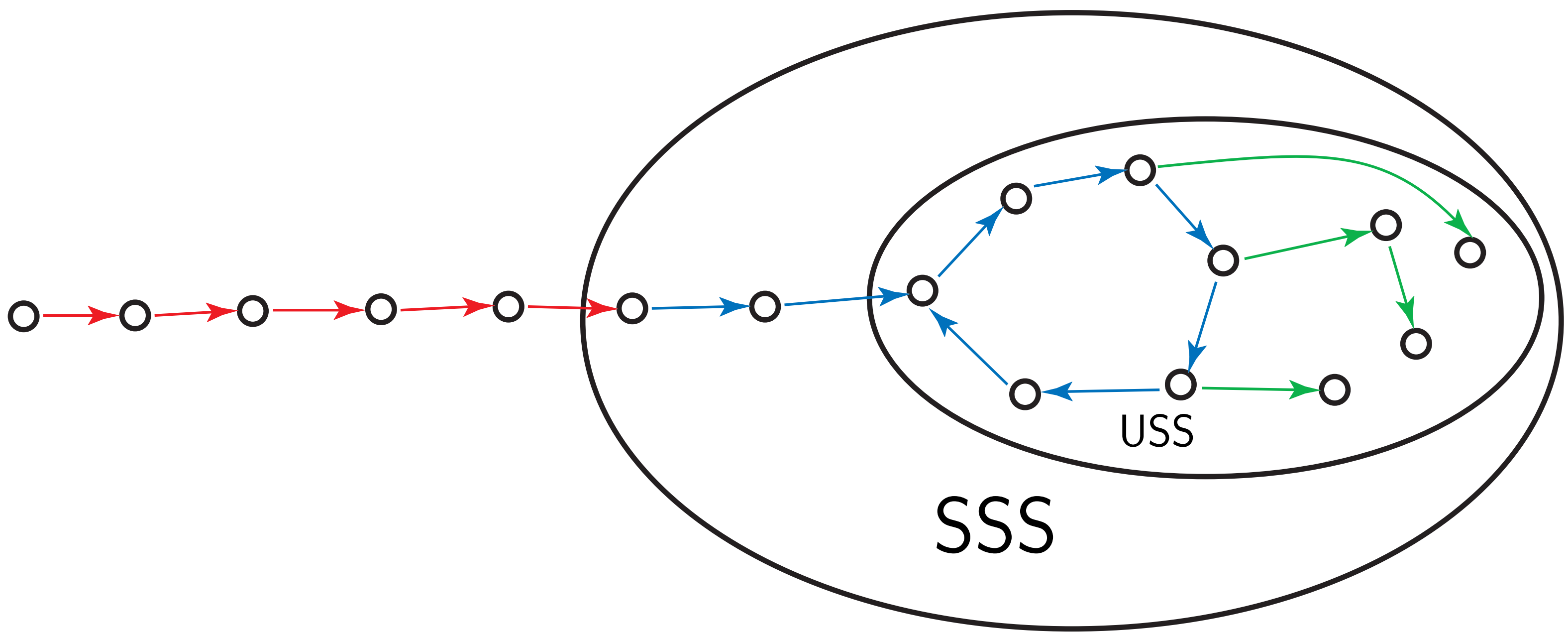
Decycling:

$$b = \Delta_n^k b_1 b_2 \ldots b_m \mapsto \Delta_n^k b_m' b_1 b_2 \ldots b_{m-1},$$

where $b_m' = \Delta_n^{-k} b_m \Delta_n^k$.

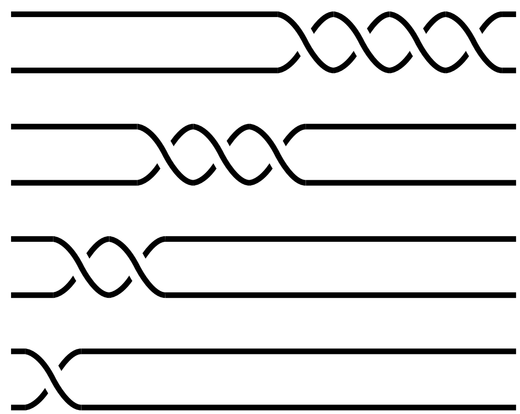
The algorithm:

- apply cycling/decycling until the braid is in SSS;
- apply cycling until a circuit is detected;
- apply minimal simple conjugations to discover the whole USS.



$|\mathsf{USS}|$ can be exponentially large. E.g., for $b_k = \sigma_1 \sigma_3^2 \ldots \sigma_{2k-1}^k \in B_{2k}$ we have

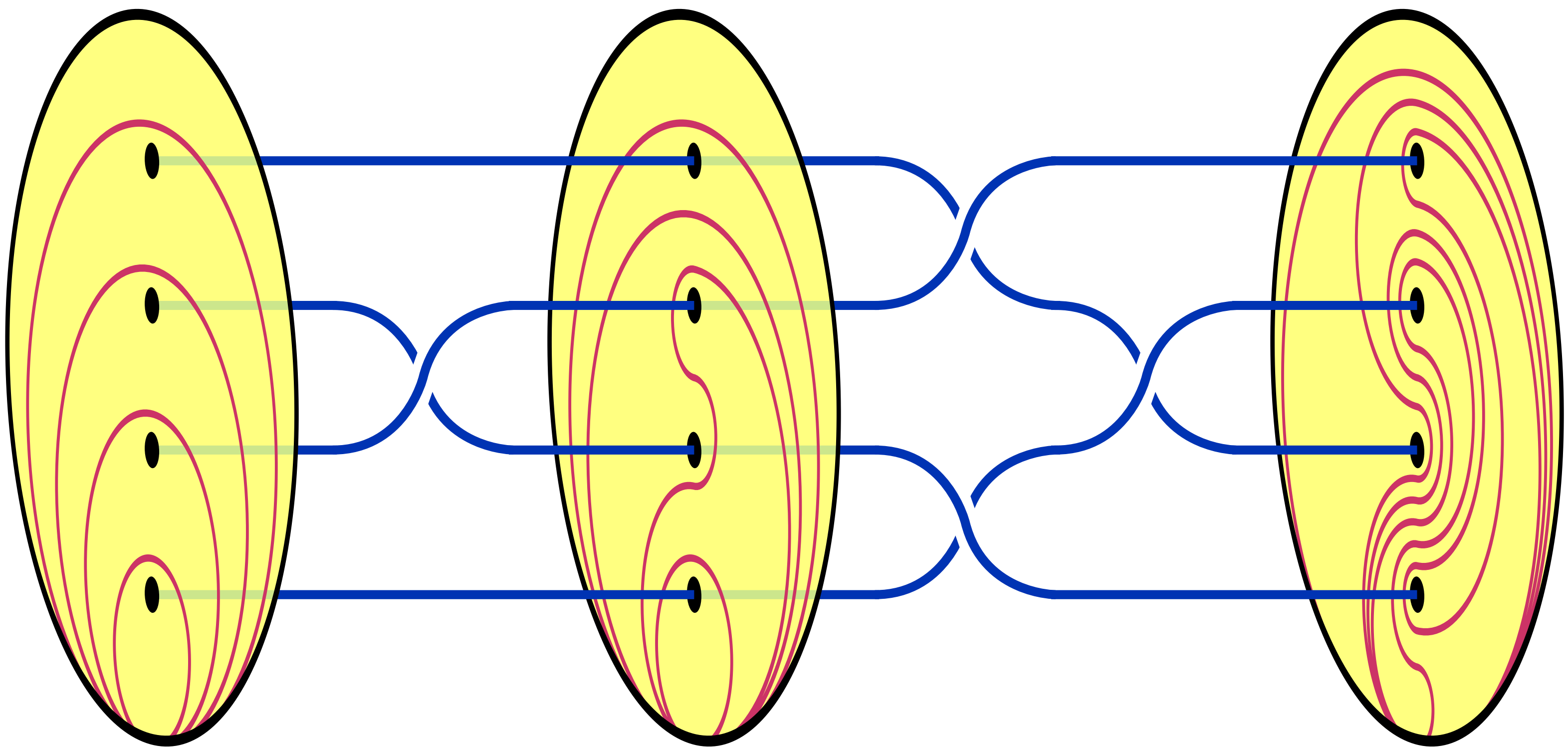
$$|\mathsf{USS}(b_k)| = k!.$$



The reason here is the reducibility of the braids.

Geometric point of view:

$$B_n \cong \mathcal{MCG}(D^2 \setminus \{P_1, \ldots, P_n\}; \partial D^2)$$

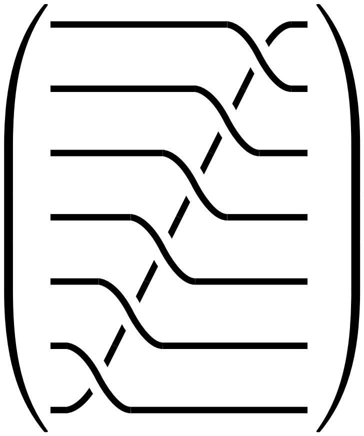


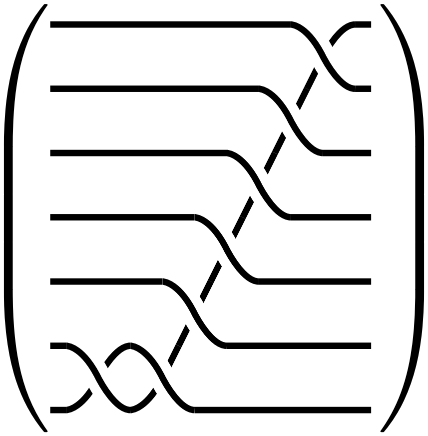
$$B_n/\langle \Delta^2 \rangle \cong \mathcal{MCG}(S^2 \setminus \{P_0 = \infty, P_1, \ldots, P_n\})$$

Nielsen–Thurston trichotomy in braid groups:

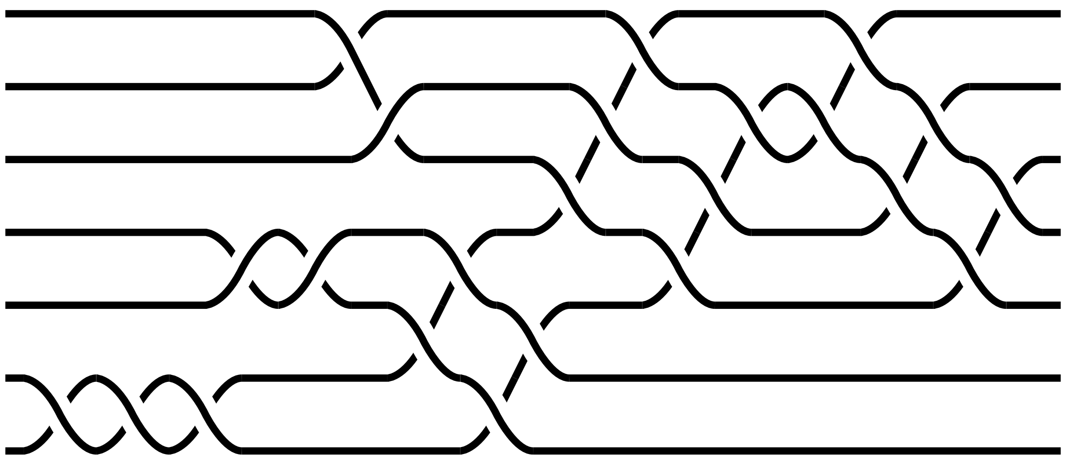
- Periodic
- Reduced
- Pseudo-Anosov

Periodic braids = roots of central elements:

c  ^{k} c^{-1}

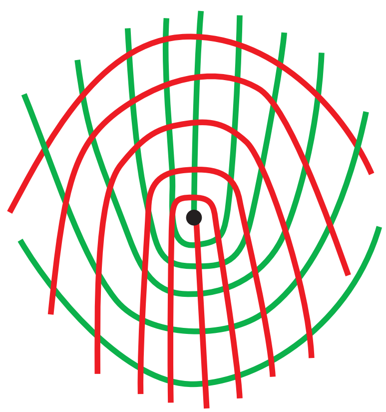
c  ^{k} c^{-1}

A reduced braid:

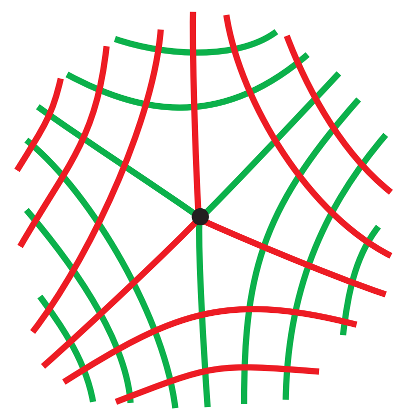


Pseudo-Anosov braid: $\exists$ two invariant mutually transversal measured foliations $\mathcal{F}_1$, $\mathcal{F}_2$ (called stable and unstable, respectively) with isolated singularities on $S^2 \setminus \{P_0, \ldots, P_n\}$, and $\lambda > 1$ s.t.

- 1-prong singularities may occur only at the punctures and $P_0 = \infty$;
- the transversal measure of $\mathcal{F}_2$ stretches λ times and that of $\mathcal{F}_1$ shrinks λ times under the action of the braid.



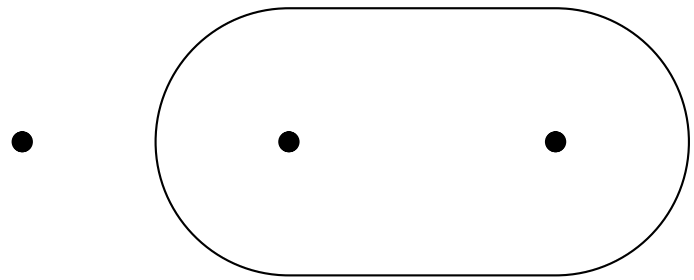
1-prong singularity



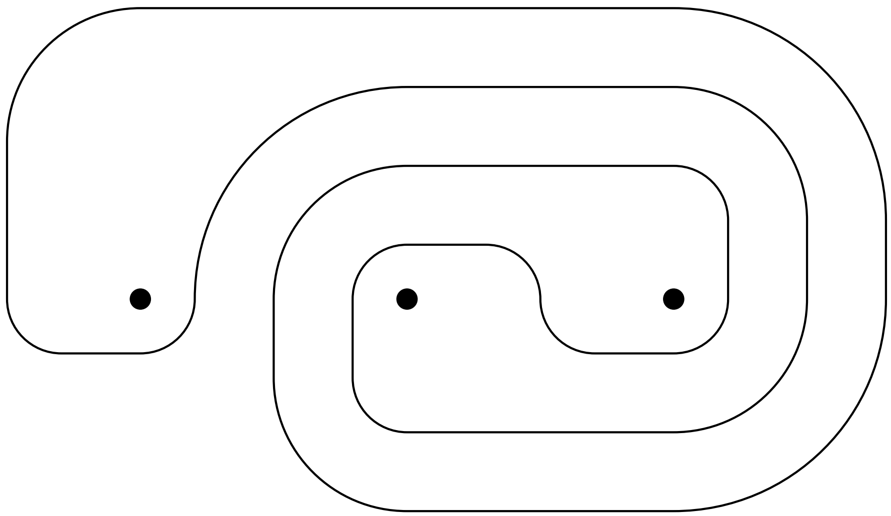
3-prong singularity

A pseudo-Anosov braid:

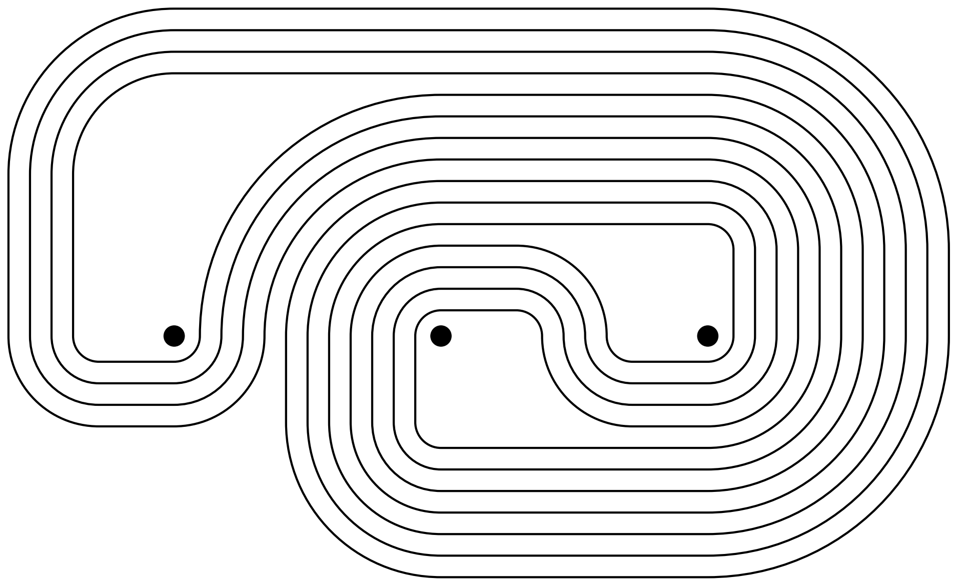
To see how $\mathcal{F}_1$ looks like one can pick an arbitrary curve linked with the punctures:



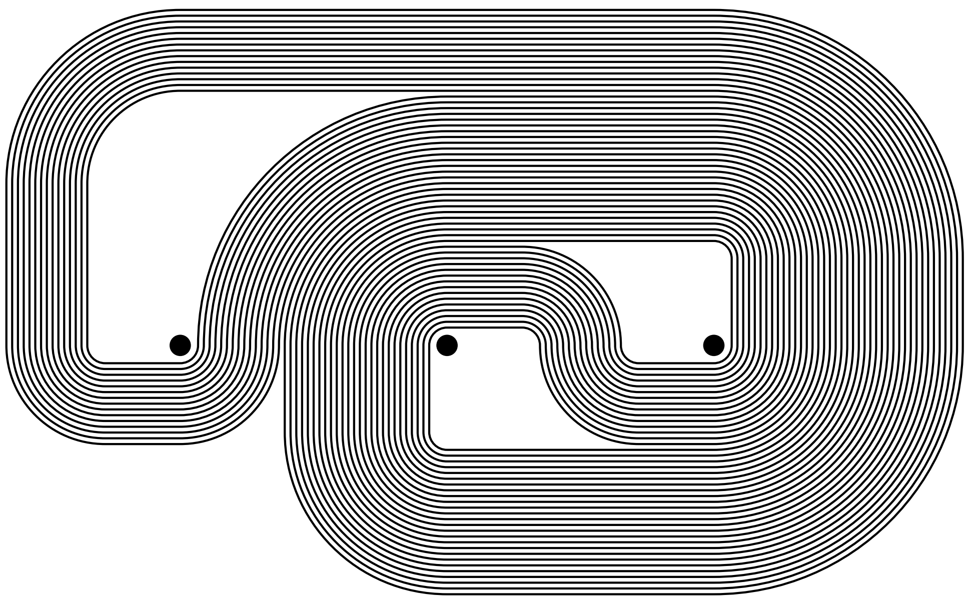
and apply a large enough power of the braid:



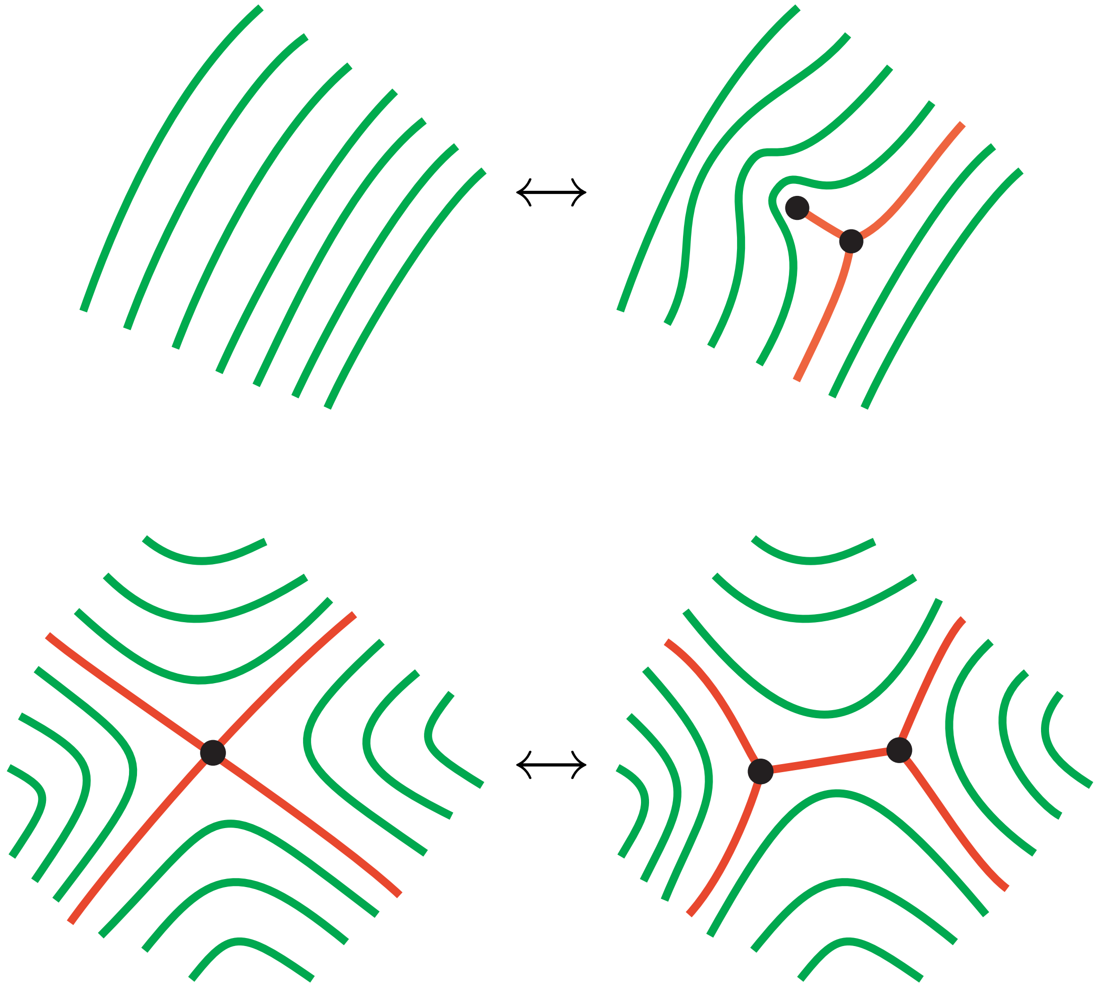
and apply a large enough power of the braid:



and apply a large enough power of the braid:



The "limit point" is equivalent to $\mathcal{F}_1$ modulo the following operations:



J.Birman, V.Gebhardt, J.Gonzales-Meneses, 2007: a polynomial solution for the Conjugator Search problem in the periodic case.

M.Bestvina, M.Handel, 1995: an algorithm for fining the geometrical type of a braid (more generally, of a surface homeomorphism). Fast in practice. Not proven to be polynomial.

The most important case is pseudo-Anosov.

A typical braid is pseudo-Anosov, it's USS consists of just one or two circuits of length bounded by the length of the braid, and all braids in the USS are rigid.

J.Birman, V.Gebhardt, J.Gonzáles-Meneses
arXiv:math/math.GT/0605230:
A small (bounded by a polynomial in n) power of a pseudo-Anosov braid has USS consisting of rigid elements.

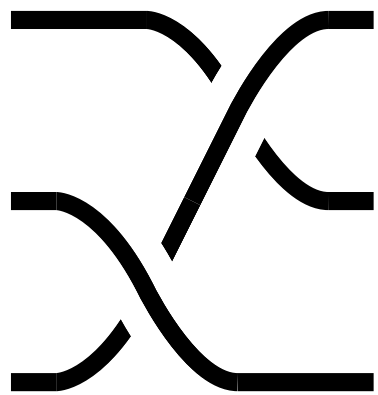
QUESTION: is there a polynomial upper bound on the size of the USS of a pseudo-Anosov rigid braid?

A typical braid is pseudo-Anosov, it's USS consists of just one or two circuits of length bounded by the length of the braid, and all braids in the USS are rigid.

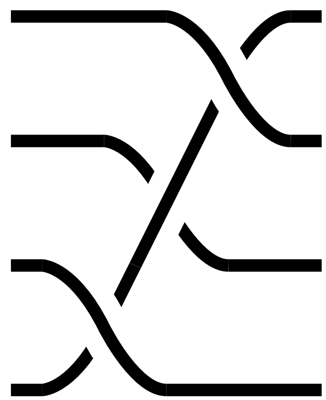
J.Birman, V.Gebhardt, J.Gonzáles-Meneses arXiv:math/math.GT/0605230:
A small (bounded by a polynomial in n) power of a pseudo-Anosov braid has USS consisting of rigid elements.

QUESTION: is there a polynomial upper bound on the size of the USS of a pseudo-Anosov rigid braid?

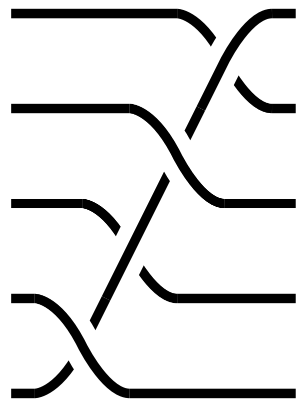
ANSWER: no.



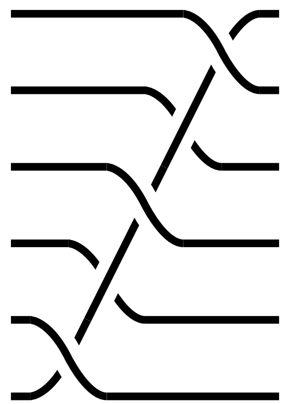
$$|\mathsf{USS}| = 4$$



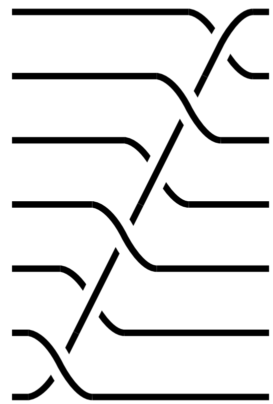
$$|\mathsf{USS}| = 6$$



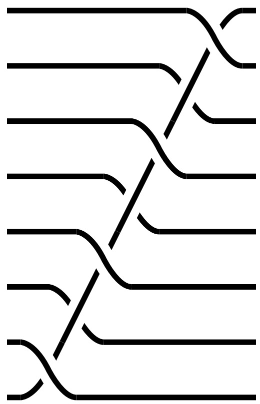
$$|\mathsf{USS}| = 36$$



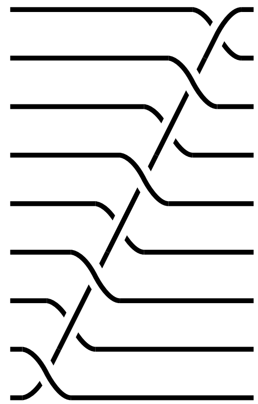
$$|\mathsf{USS}| = 54$$



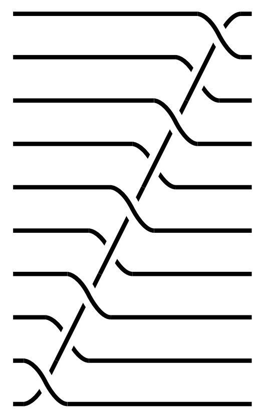
$$|\mathsf{USS}| = 324$$



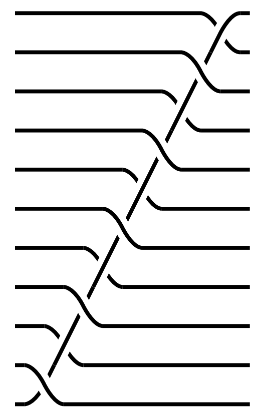
$$|\mathsf{USS}| = 486$$



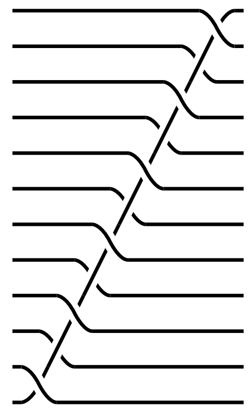
$$|\mathsf{USS}| = 2916$$



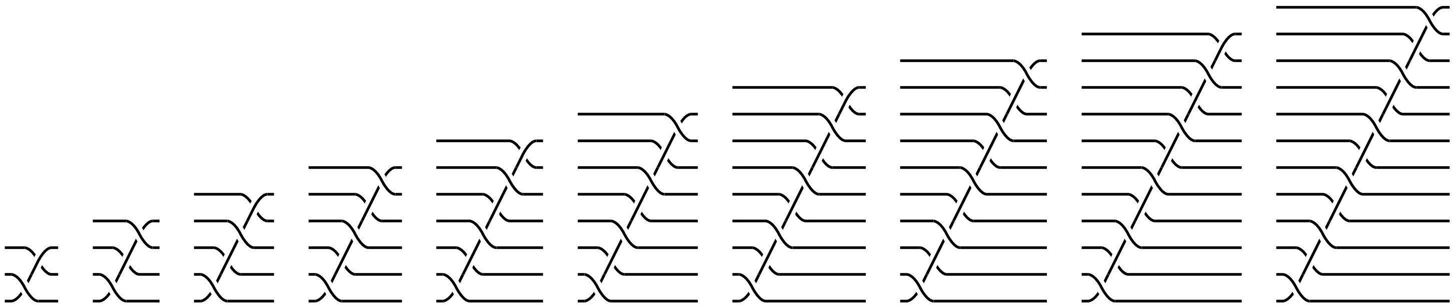
$$|\mathsf{USS}| = 4374$$



$$|\mathsf{USS}| = 26244$$



$$|\mathsf{USS}| = 39366$$

braid										
$\lvert\mathrm{USS}\rvert$	4	6	36	54	324	486	2916	4374	26244	39366

Conjectured formula: $(3 + (-1)^{n-1}) \cdot 3^{n-3}$

M.Prasolov: $|\mathrm{USS}| \geqslant 2^{n/2-1}$ in this case.

Birman–Ko–Lee setup.

Conjectured formula for |USS| of $\sigma_1\sigma_2^{-1}\dots\sigma_{n-1}^{(-1)^n}$ (holds for small n):

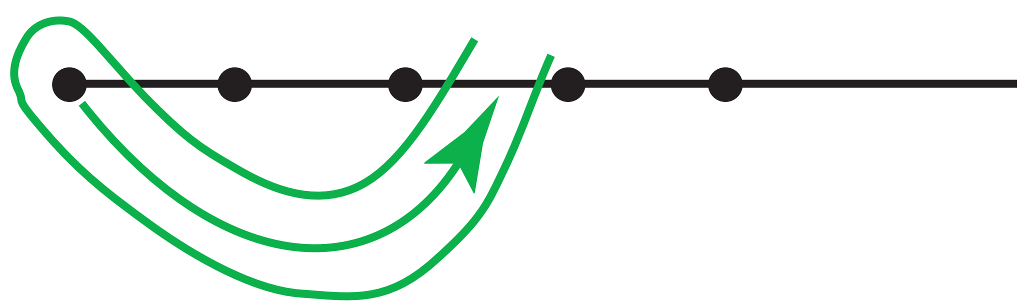
$$\begin{cases} 2n\cdot 3^{n-3}, & n \text{ odd},\\ n\cdot 3^{n-3}, & n \text{ even}.\end{cases}$$

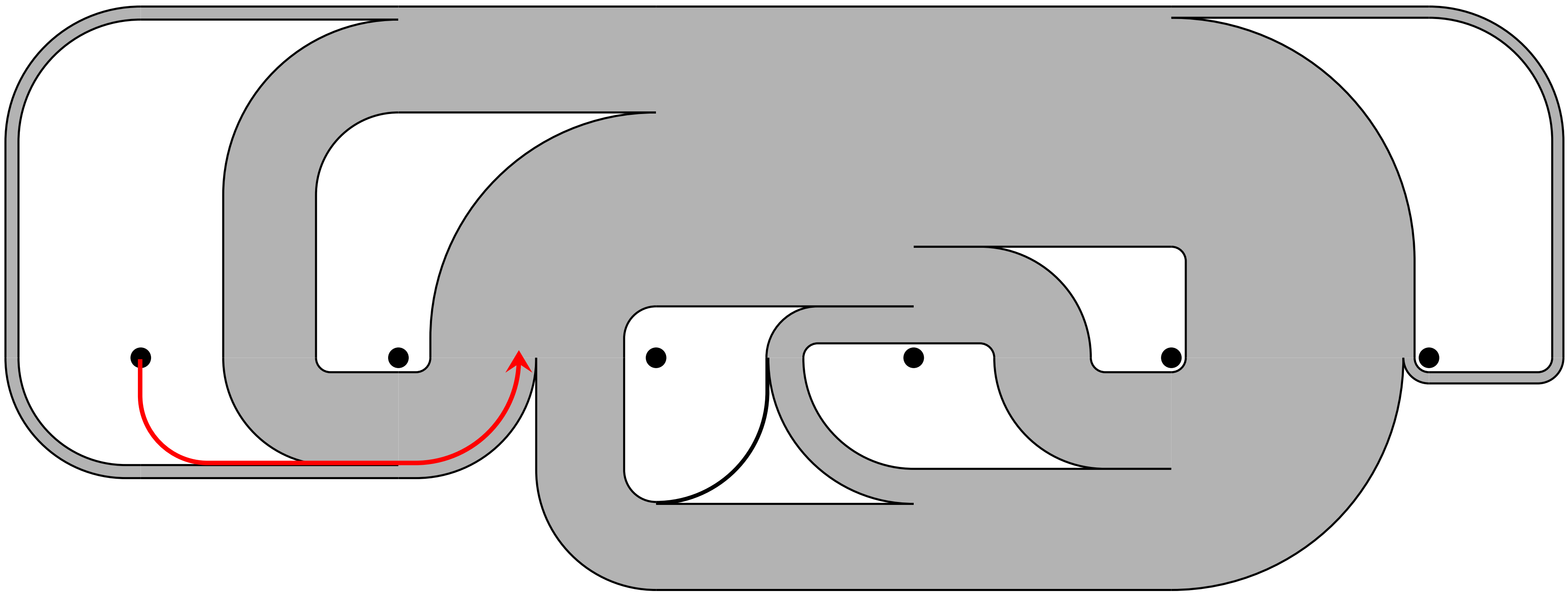
Proven growth for odd n (M.Prasolov): $2^{(n-1)/2}$.

Geometric cycling for a pseudo-Anosov braid:

$$b \mapsto a^{-1}ba,$$

where a corresponds to moving the leftmost puncture P_1 along a separatrix of $\mathcal{F}_1$ to the leftmost available position on the ray $[P_1, \infty)$

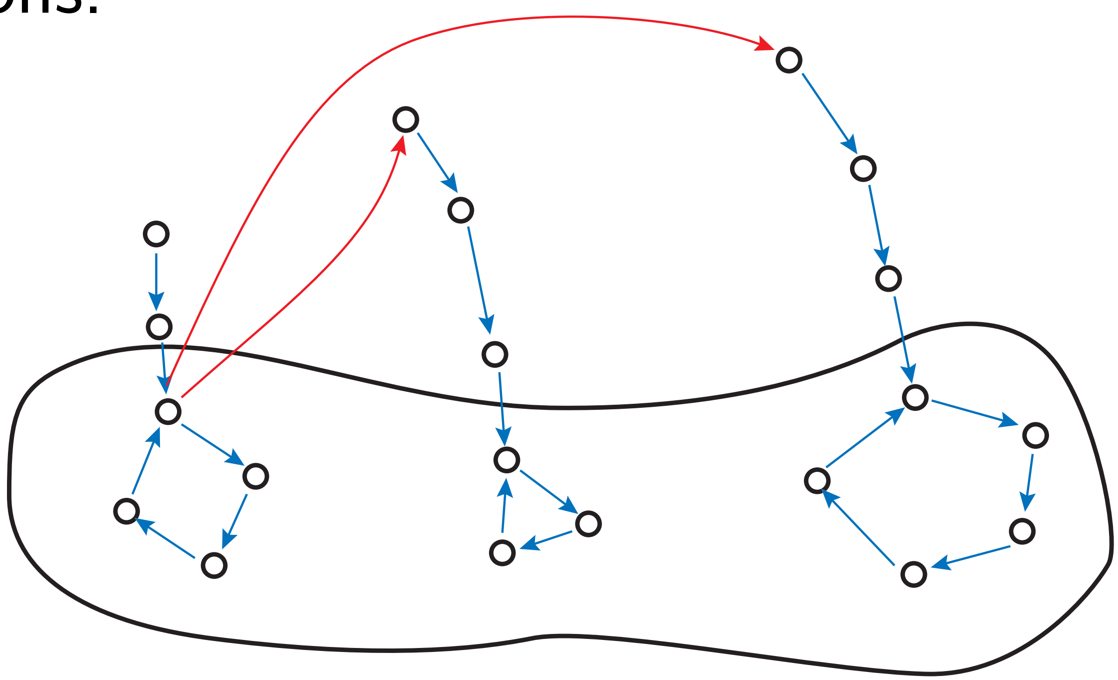




The algorithm:

$(*)$ apply geometric cycling until finding a circuit;

- apply elementary conjugations to an element from the circuit;
- repeat $(*)$ for each of the braid obtained by the elementary conjutations.

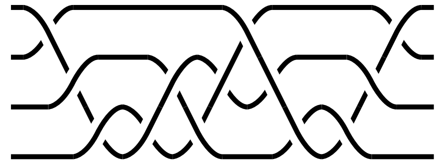


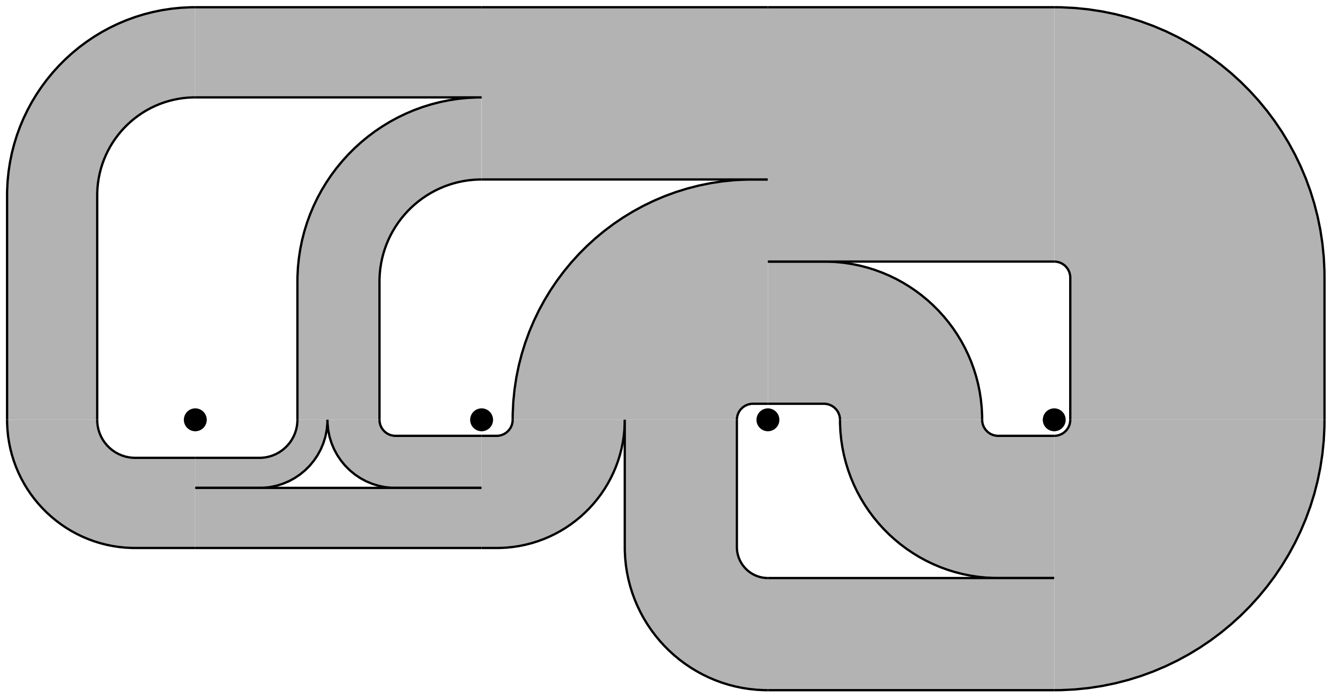
Geometrical summit set contains at most $(n-2)$ circuits!

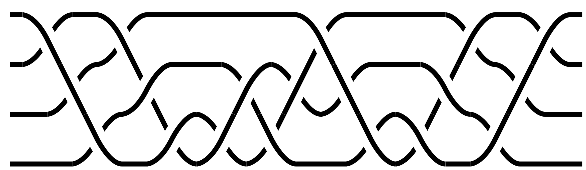
More precisely, the number of circuits equals k/q, where:
k is the number of prongs at the infinity;
q is the denominator of the rotation number $\rho = p/q$ with $\gcd(p,q) = 1$, $q > 0$. The rotation number of a braid is characterized by the property

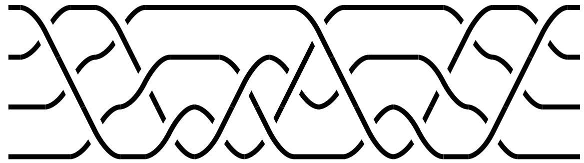
$$\Delta^{2[m\rho]-2} < b^m < \Delta^{2[m\rho]+4},$$

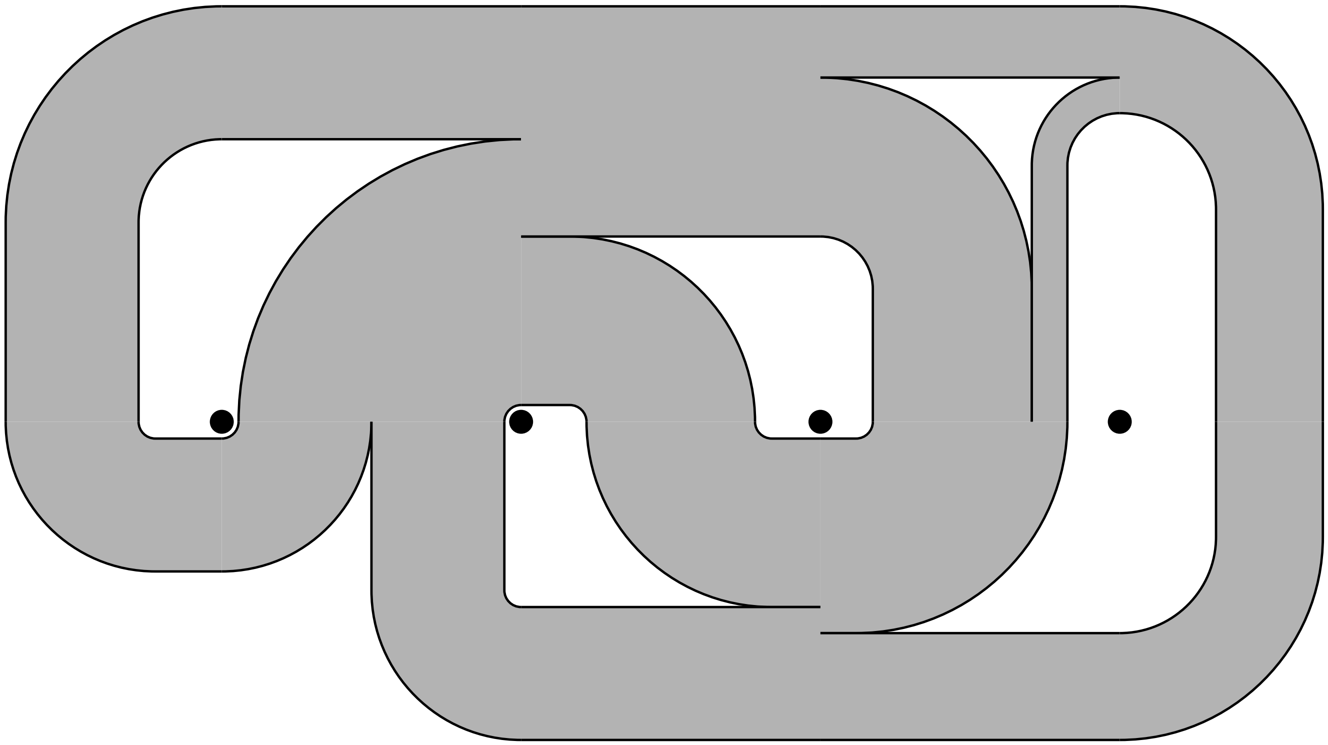
which holds for any $m \in \mathbb{Z}$ and Dehornoy's (more general, any Thurston type) ordering "$<$".

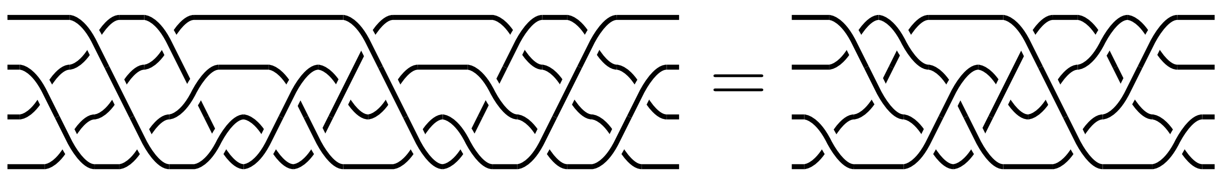


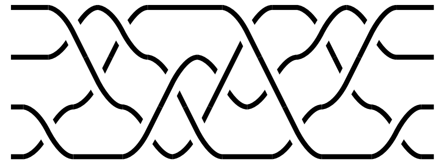


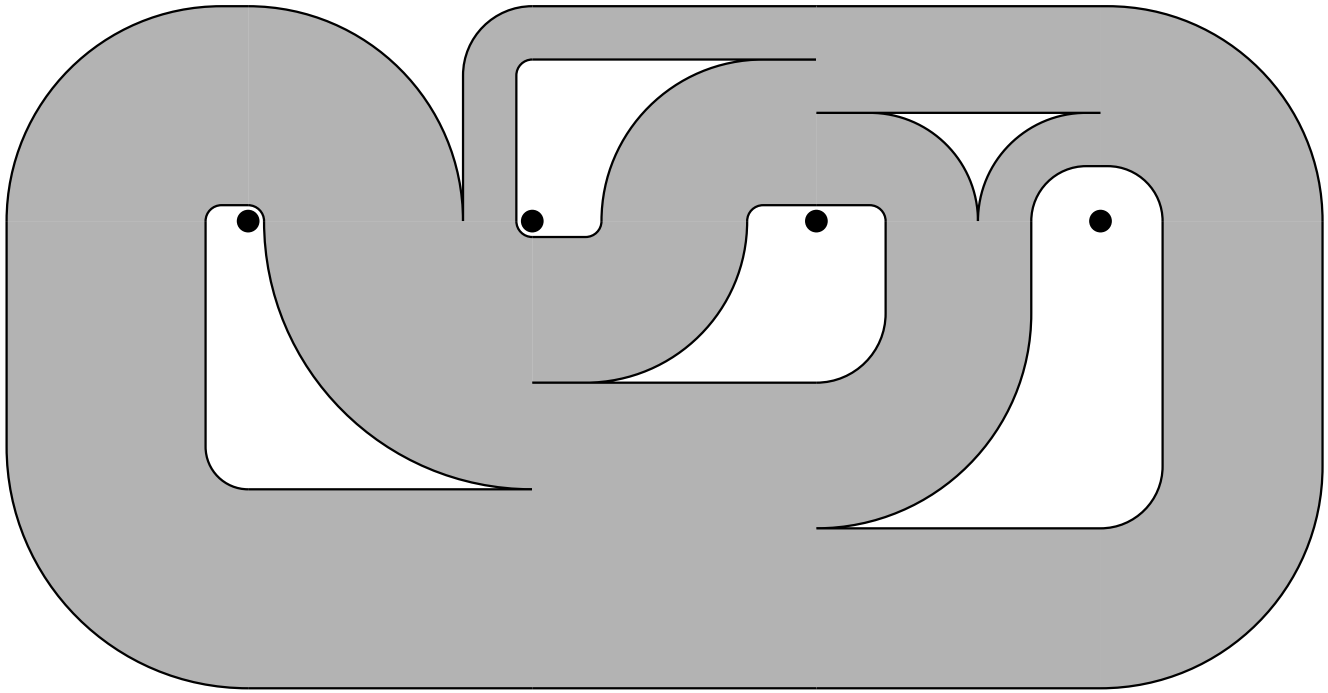


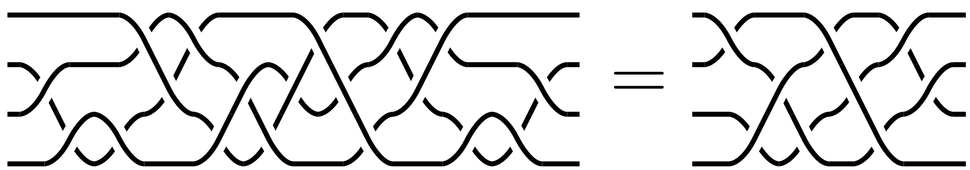


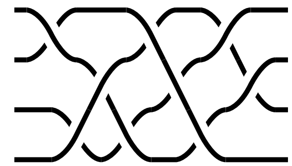


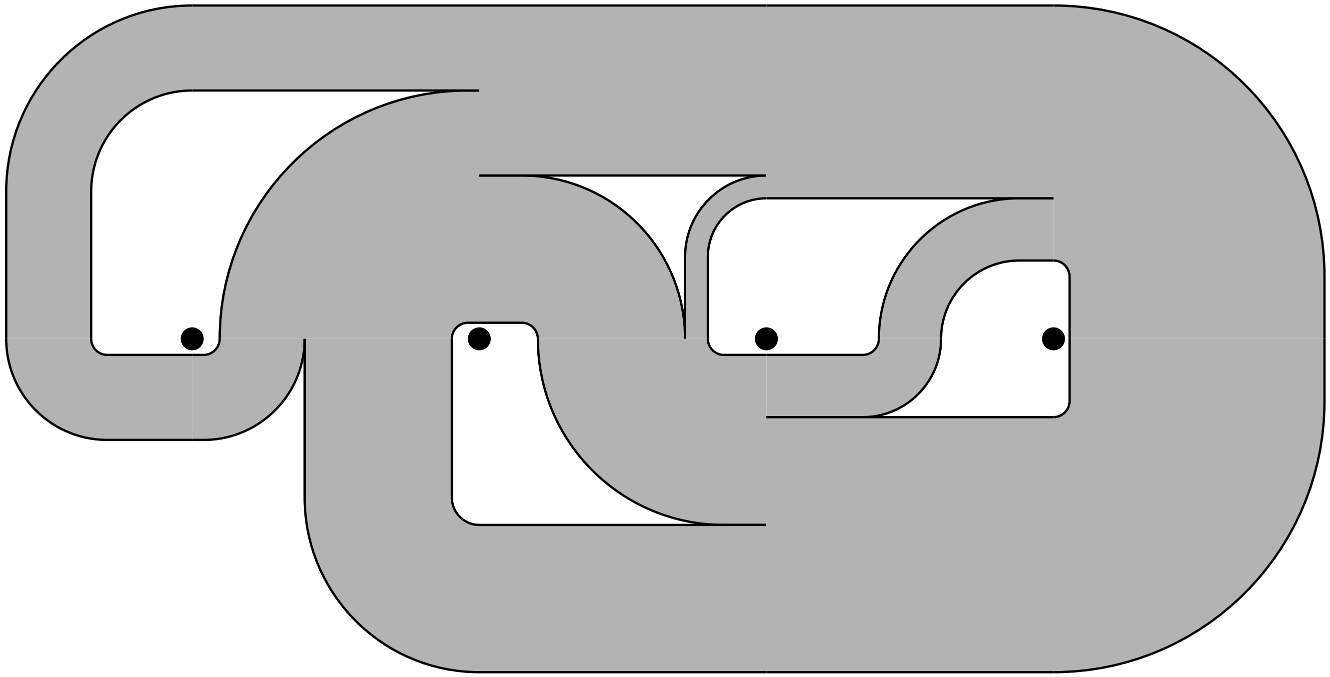


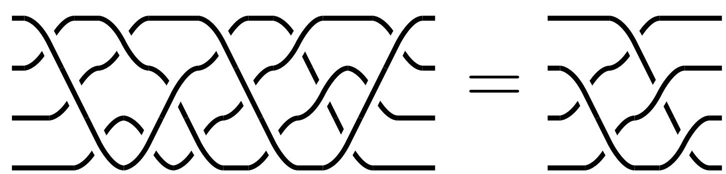


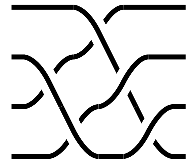


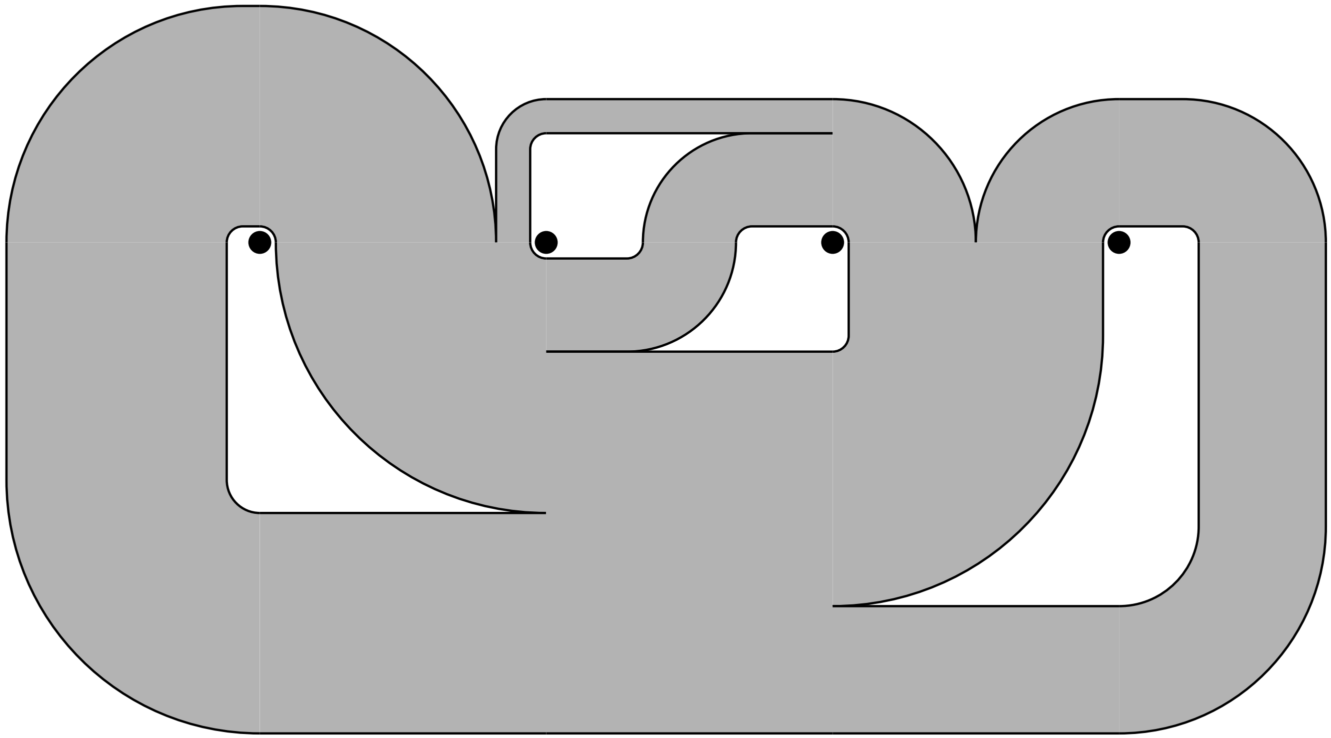


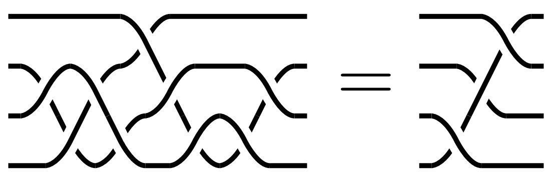


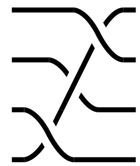


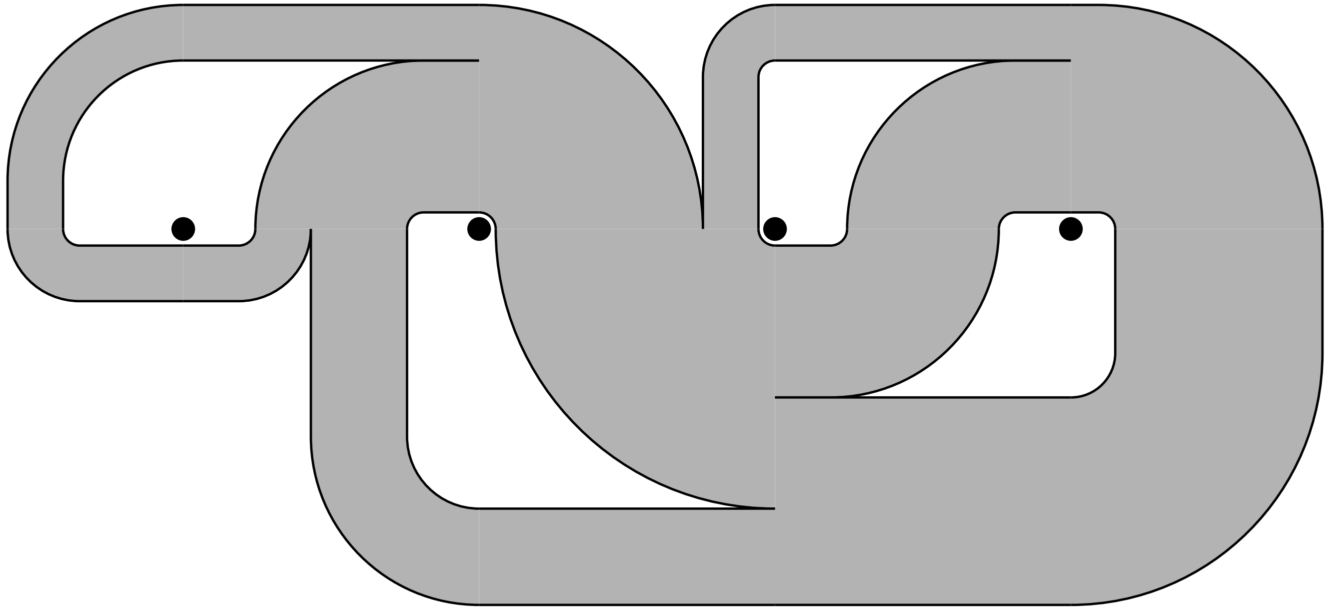


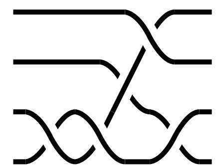


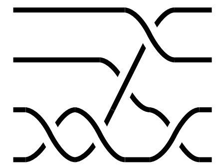


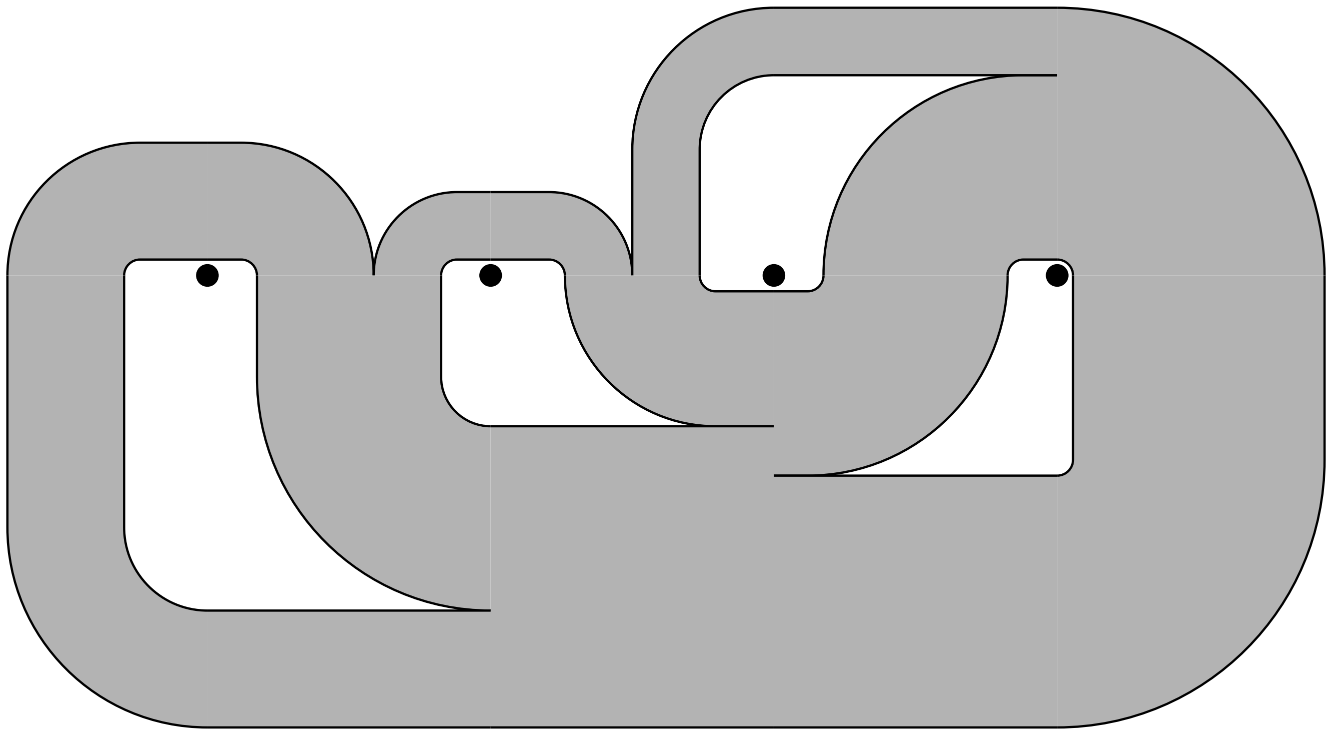


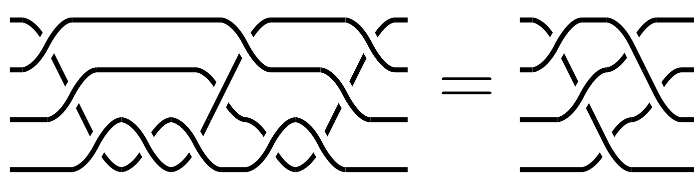


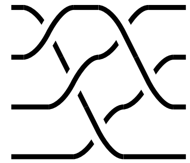


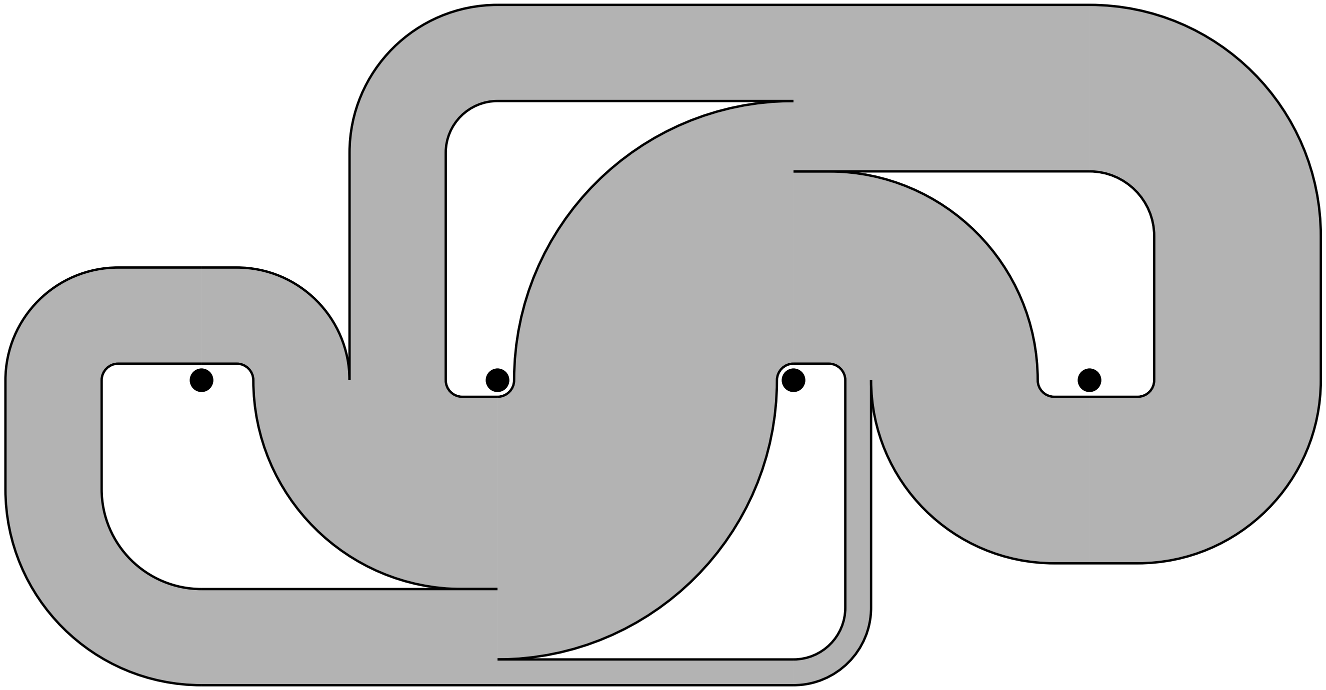


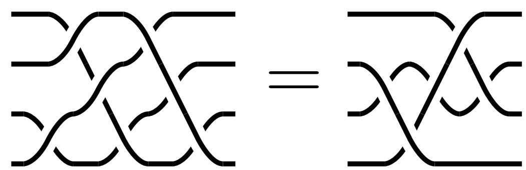


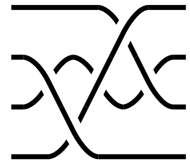


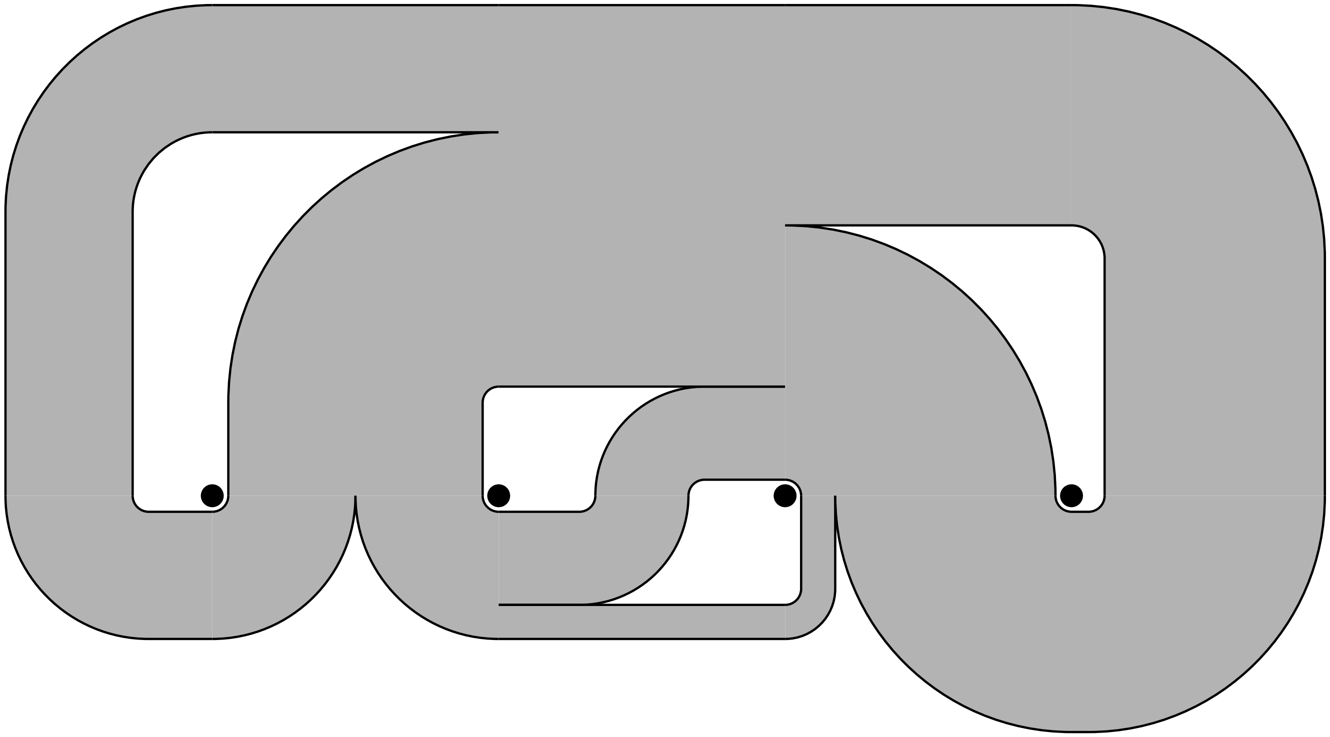


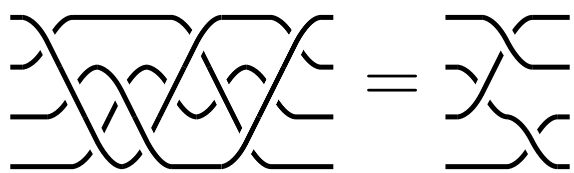


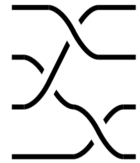


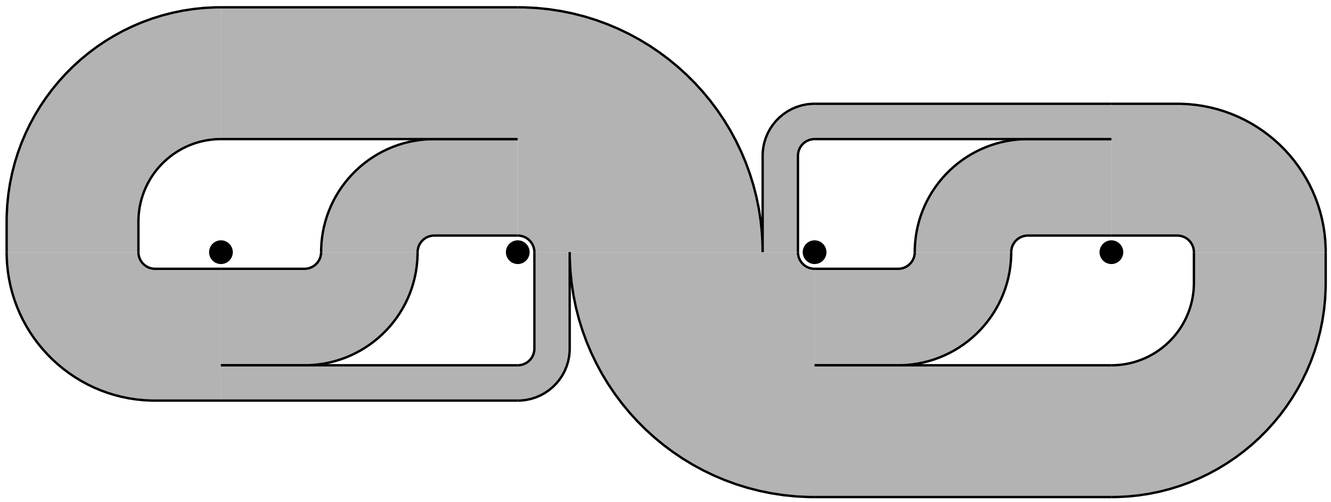


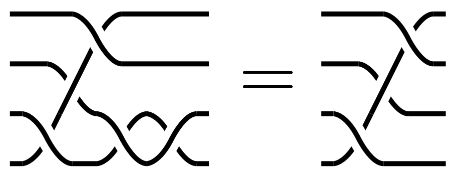


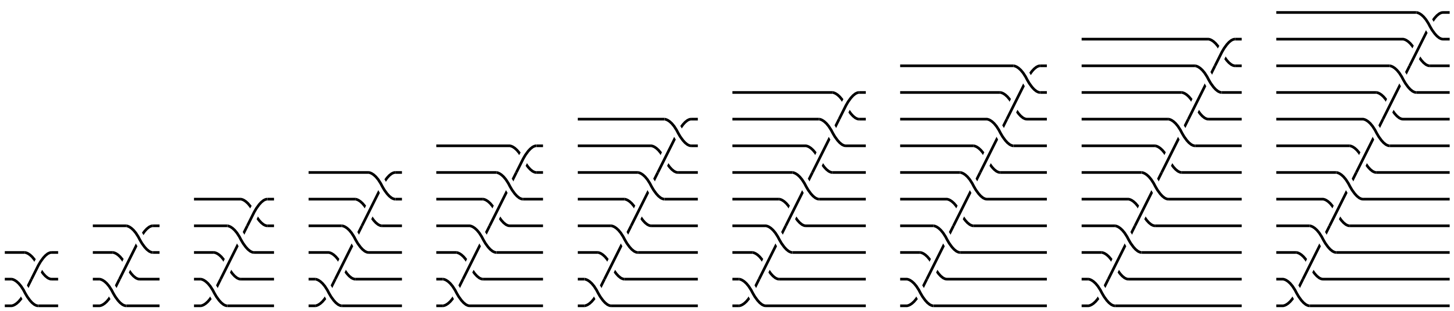












braid										
$\lvert\mathrm{USS}\rvert$	4	6	36	54	324	486	2916	4374	26244	39366
$\lvert\mathrm{GSS}\rvert$	2	5	24	24	82	65	192	136	370	245