Free subgroups of lattices

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Main Problem

Given a lattice Γ in a locally compact unimodular group G, prove the existence of subgroups of Γ satisfying prescribed conditions.

Strategy

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- the isomorphism type of F' is close to that of F,
- large-scale properties of $F' < G$ are close to large-scale properties of $F < G$,
- $F' < \Gamma$?

Example

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After a small algebraic deformation of F, we may assume τ is rational.

Then, a finite-index subgroup of F lies in Γ.

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$\phi_\epsilon : F \to G$ is an *ϵ-perturbation* of ϕ if for any sequence $s_1, \ldots, s_n \in S \cup S^{-1}$, there exist elements $s'_i \in G$ such that

$$d(\phi(s_i), s'_i) < \epsilon,$$

$$\phi_\epsilon(s_1 \cdots s_i) = s'_1 \cdots s'_i$$

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Equivalently, $\forall s \in S \cup S^{-1}, f \in F$,

$$d\big(\phi_\epsilon(fs), \phi_\epsilon(f)\phi(s)\big) < \epsilon.$$

$\phi_\epsilon : F \to G$ is *virtually a homomorphism* into $\Gamma < G$ if $\exists$ a finite index subgroup $F' < F$ such that

$$\phi_\epsilon(f_1)\phi_\epsilon(f_2) = \phi_\epsilon(f_1 f_2) \; \forall f_1 \in F', f_2 \in F$$

and $\phi_\epsilon(F') < \Gamma$.

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- If $\phi_\epsilon(id) = id$ then $\lim_{\epsilon\to 0} L(\phi_\epsilon) = L(\phi)$.
- If ϕ_ϵ is a virtual homomorphism then $\lim_{\epsilon\to 0} H.dim\ L(\phi_\epsilon) = H.dim\ L(\phi)$ as $\epsilon \to 0$.

H. Dimensions of Limit Sets of Free Subgroups

For $\Lambda < G$, let $D_{free}(\Lambda) = \{d \geq 0 : \exists$ quasi-convex cocompact free subgroup $F < \Lambda$ with $H.dim\ L(F) = d\}$.

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Remark

If $G = Isom(\mathbb{H}^n)$ for $n = 2$ or 3 then $\overline{D_{free}(G)} = [0, n-1]$. No nontrivial bounds are known for $n \geq 4$.

The Cheeger Constant

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Theorem (Lackenby, Long, Reid)

If M is a closed hyperbolic 3-manifold then there exists a sequence of infinite-sheeted coverings M_i of M such that $h(M_i) \to 0$.

LERF and Property τ

A group Γ is *LERF* $\iff$ $\forall$ finitely generated subgroup $H < \Gamma$,

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I.e., the Lubotzky-Sarnak conjecture holds for $\pi_1(M)$.

Surface Subgroups

Theorem (Lackenby)

If $\Gamma < SO(3,1)$ is discrete, finitely generated and contains a noncyclic finite subgroup then either Γ is finite, Γ is virtually free or Γ contains a surface subgroup.

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The new system has a periodic point.

The periodic point corresponds to the ϵ-perturbation that we need.

Symbolic dynamics

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The *shift map* of $f \in F$ is the homeomorphism $\sigma_f : K^F \to K^F$ defined by

$$\sigma_f x(g) = x(f^{-1}g).$$

Finite-Type Constraints

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$\mathcal{G}$ is a *constraint graph*.

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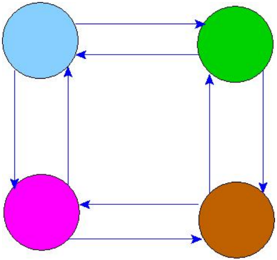

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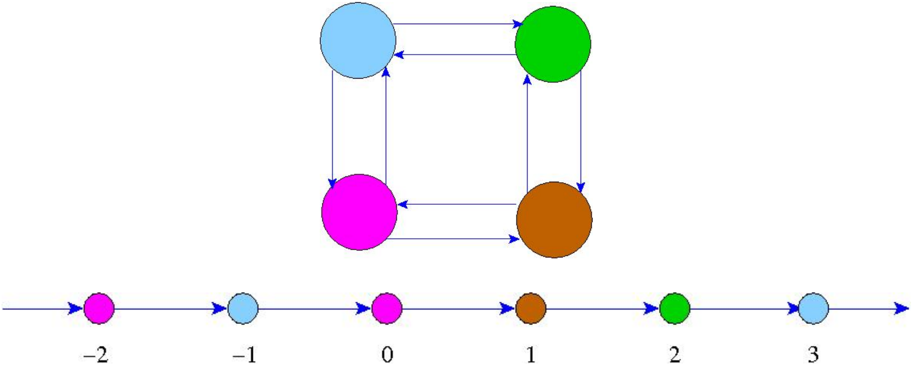
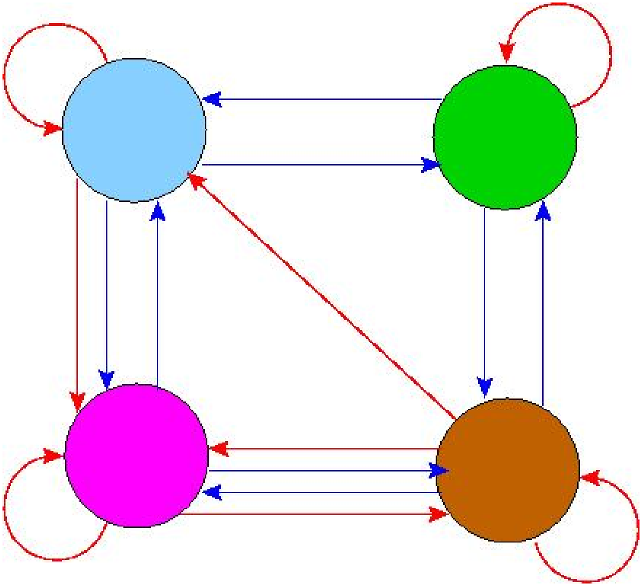
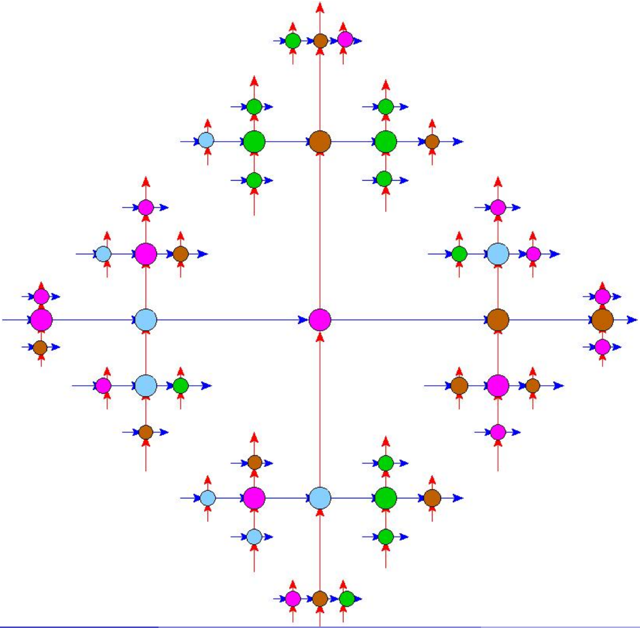


A constraint graph for $F = \mathbb{F}_2$



The graph-shift determined by $\mathcal{G}$



The graph-shift determined by $\mathcal{G}$

Let $X = \{x \in K^F \mid \forall g \in F, s \in S$, if $x(g) = i$ and $x(gs) = j$ then $\exists$ an edge in $\mathcal{G}$ from i to j labeled $s\}$.

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Lemma (Key Lemma)

Let $X \subset K^F$ be a subshift of finite type.
If $\exists$ a shift-invariant Borel probability measure on X then $\exists$ a periodic point in X. Indeed, invariant measures supported on periodic points are dense in the space of all invariant measures on X.

Proof of main theorem given key lemma

Let $\delta > 0$ be such that $\forall g_1, g_2 \in G$ with $d(g_1, id), d(g_2, id) < \delta$ and $\forall s \in S \cup S^{-1}$,

$$d\big(g_1\phi(s)g_2, \phi(s)\big) < \epsilon.$$

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Let $A_1, \ldots, A_n$ be a Borel partition of $\Gamma\backslash G$ into sets of diameter $< \delta$.

Choose $a_i \in A_i$. Assume $a_1 = \Gamma$, the identity coset.

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$\forall a_i, a_j \in V(\mathcal{G})$, $\forall s \in S$, $\exists$ an s-labeled edge $e = (a_i, a_j; s)$ from a_i to a_j iff $\exists a_i' \in A_i, a_j' \in A_j$ such that $a_i' \phi(s) = a_j'$.

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Define $\psi(e) = g_i \phi(s) g_j^{-1}$.

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Note $a_i \psi(e) = a_j$ and $d\big(\psi(e), \phi(s)\big) < \epsilon$.

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For $x \in X$ and $f \in F$, define $\phi_x(f) = \psi_x(e_1) \cdots \psi_x(e_n)$ where $e_1, \ldots e_n$ is the unique path in the Cayley graph of F from *id* to f.

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ϕ_x is an ϵ-perturbation of ϕ.

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From periodic points to finite graphs

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A periodic point $x \in K^F$ corresponds to a finite labeled graph (the Schreier coset graph of Stab(x)).

From periodic points to finite graphs

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Conversely, a finite appropriately labeled graph corresponds to a periodic point.

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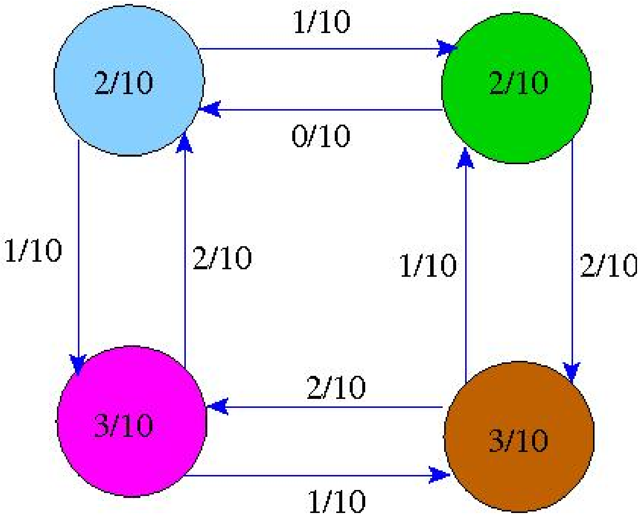

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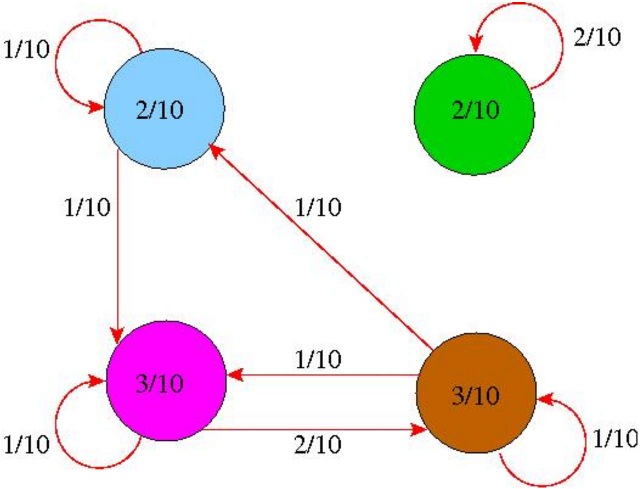

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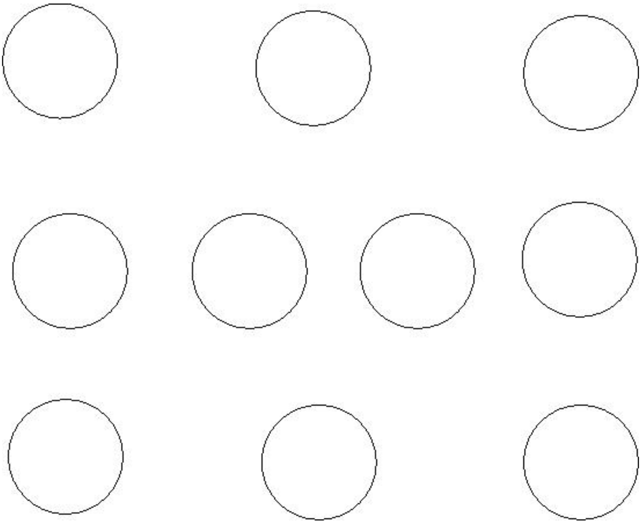
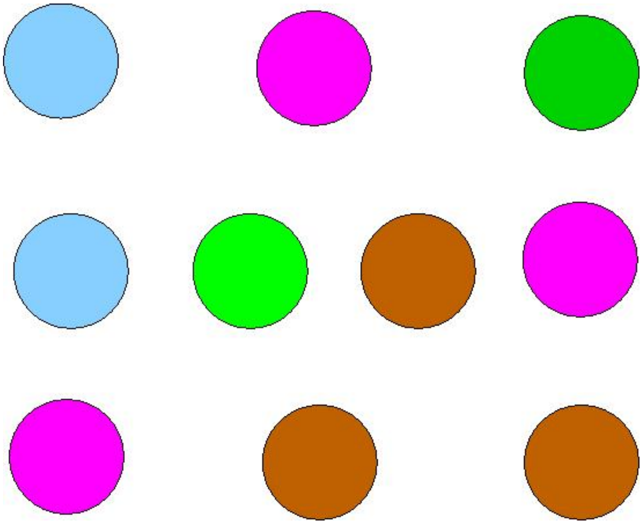
Conversely, from a nonzero integral W there exists a finite graph $\mathcal{H}$ (with universal cover the Cayley graph of F) with $W_{\mathcal{H}} = W$.

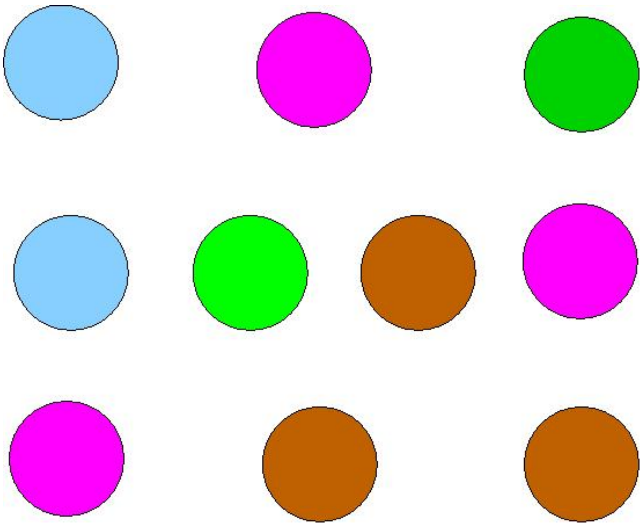
How to construct the finite graph: step 1



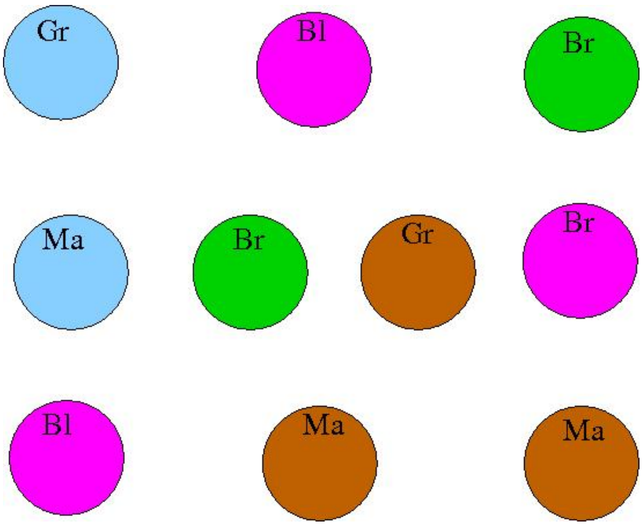
How to construct the finite graph: step 2



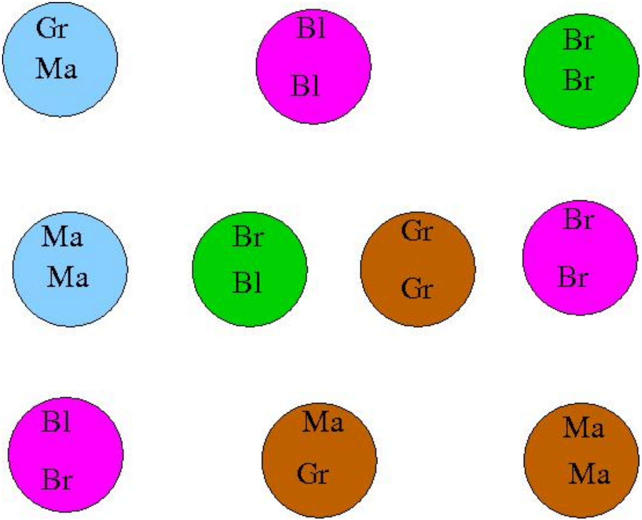
How to construct the finite graph: step 3



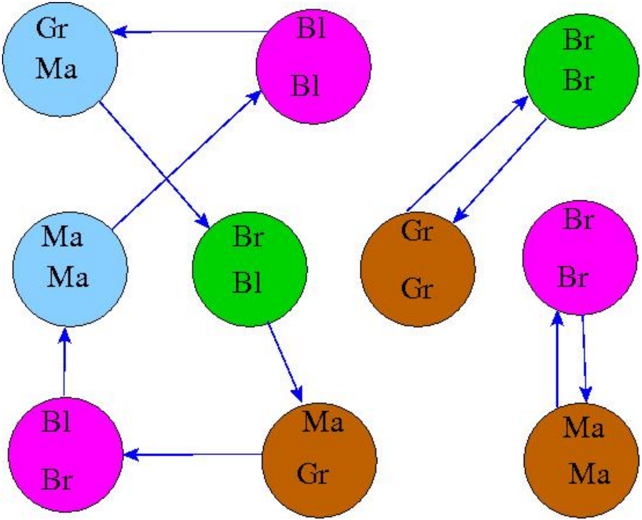
How to construct the finite graph: step 4



How to construct the finite graph: step 5



How to construct the finite graph: step 6



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Surface Groups

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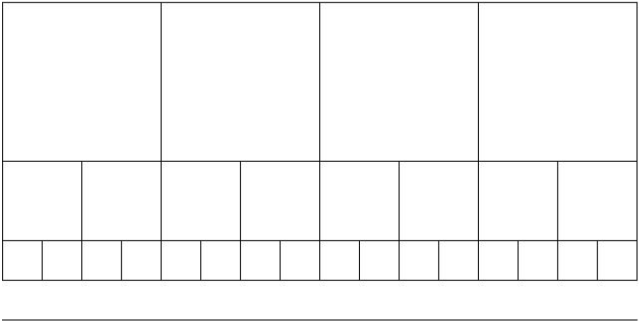
Continuous version : There is a finite set of tiles in the hyperbolic plane such that no periodic tiling with these tiles exists but there is an $Isom(\mathbb{H}^2)$-invariant probability measure on the space of tilings.

An aperiodic tile set



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