Automata generating free products of groups of order 2

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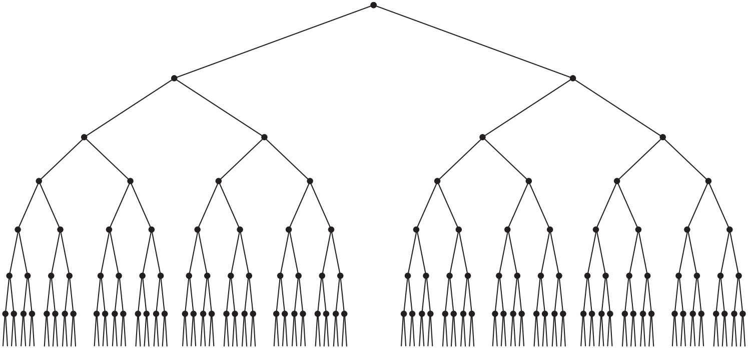
The space we act on

Action on a rooted tree T.
$V(T) = X^*, \qquad X = \{0, \ldots, d-1\}$ – alphabet

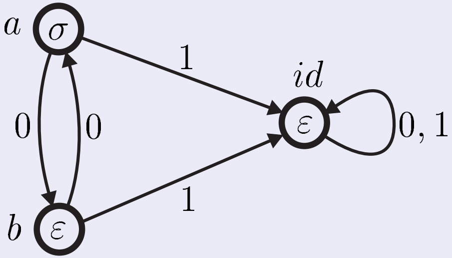


$$G < \mathrm{Aut}\, T$$

Action given by finite Mealy type automaton

Definition (By Example)

$S_2 = \{\varepsilon = id, \sigma = (01)\}$ acts on $X = \{0, 1\}$.



$\mathcal{A}$ — noninitial automaton,
$\mathcal{A}_q$ — initial automaton, $q \in \{a, b, id\}$.

$\mathcal{A}_q$ acts on X^* (and on T)

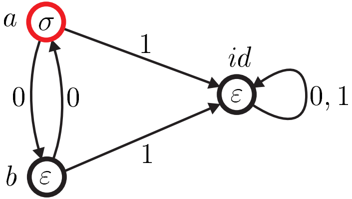
Input:	0	0	0	0	1	0	1	1

$\downarrow$

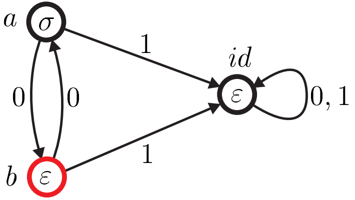
States:	a	b	a	b	a	id	id	id

$\downarrow$

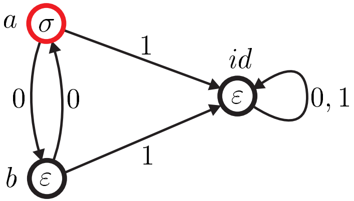
Output:	1	0	1	0	0	0	1	1



Input:	0	0	0	0	1	0	1	1
States:	a	b	a	b	a	id	id	id
Output:	1	0	1	0	0	0	1	1



Input:	0	0	0	0	1	0	1	1
States:	a	b	a	b	a	id	id	id
Output:	1	0	1	0	0	0	1	1



Input:	0	0	0	0	1	0	1	1
States:	a	b	a	b	a	id	id	id
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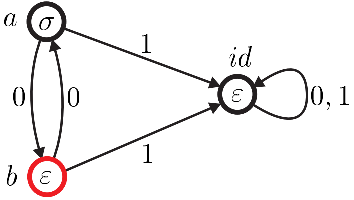

Input:	0	0	0	0	1	0	1	1
States:	a	b	a	b	a	id	id	id
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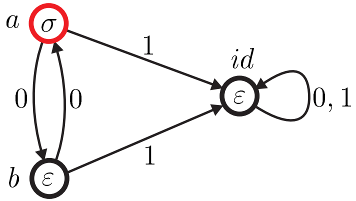

Input:	0	0	0	0	1	0	1	1
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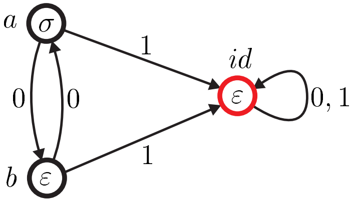

Input:	0	0	0	0	1	0	1	1
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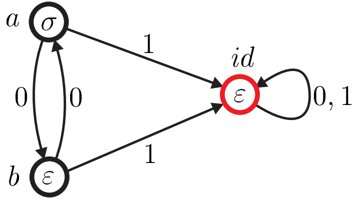

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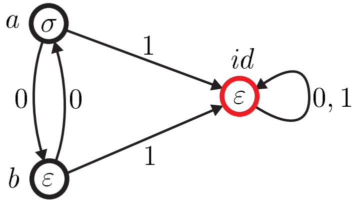

Definition of automaton group

Given an automaton A every state q defines an automorphism A_q of X^*

Definition

The *automaton* (or *self-similar*) group generated by automaton A is a group $\langle A_q | q \text{ is a state of A}\rangle < \mathrm{Aut}\, X^*$. This group is denoted by $G(A)$.

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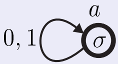
Example



$a(w) = \overline{w}$. Thus $a^2 = 1$ and $G(A) \simeq C_2$.

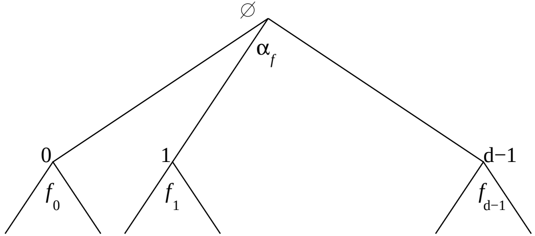
There is a convenient way to represent the element f of $\mathrm{Aut}\, X^*$ in the form

$$f = (f_0, f_1, \dots, f_{d-1})\alpha_f,$$

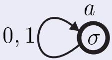
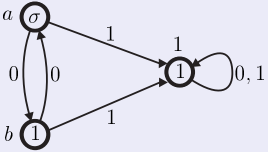
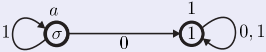
where
$f_i \in \mathrm{Aut}\, X^*$ describe how f acts on the i-th subtree, i.e.

$$f(iu) = jw \Leftrightarrow f_i(u) = w,$$

$\alpha_f \in \mathrm{Sym}(X)$ describes how f acts on the 1-st letter.

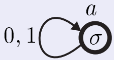


Example

C_2	Basilica	$\mathbb{Z}$ (Adding Machine)
		
$a = (a, a)\sigma$	$a = (b, 1)\sigma$ $b = (a, 1)$ $1 = (1, 1)$	$a = (1, a)\sigma$

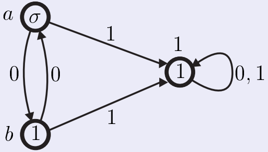
Example

C_2



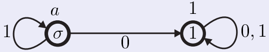
$a = (a, a)\sigma$

Basilica



$$a = (b, 1)\sigma$$
$$b = (a, 1)$$
$$1 = (1, 1)$$

$\mathbb{Z}$ (Adding Machine)



$a = (1, a)\sigma$

If

$$g = (g_1, g_2, \ldots, g_d)\pi_g,$$

$$h = (h_1, h_2, \ldots, h_d)\pi_h,$$

then

$$gh = (g_1 h_{\pi_g(1)}, \ldots, g_d h_{\pi_g(d)})\pi_g \pi_h.$$

Source of Counterexamples

- Burnside problem on infinite periodic groups
- Milnor problem on groups of intermediate growth
- Day problem on amenability
- Atiyah conjecture on L^2 Betti numbers
- Connection to holomorphic dynamics via Iterated Monodromy Groups

What known groups are generated by automata?

- $GL_n(\mathbb{Z})$
- Baumslag-Solitar groups $BS(1, n)$
- Free groups
- Free products of some groups

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- Gupta-Gupta-Oliynyk (2007) free products of finite groups

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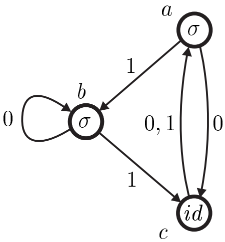
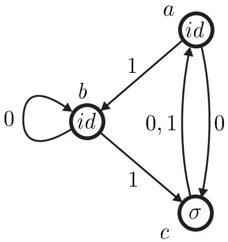


Aleshin's automaton (1983, 2005)
F_3

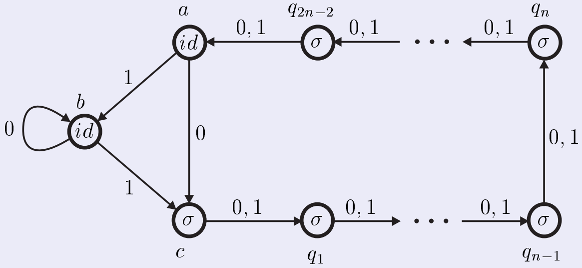


Bellaterra automaton (2004)
$C_2 * C_2 * C_2$

Generalizations

Theorem (M. Vorobets, Ya. Vorobets (2006))

The automaton

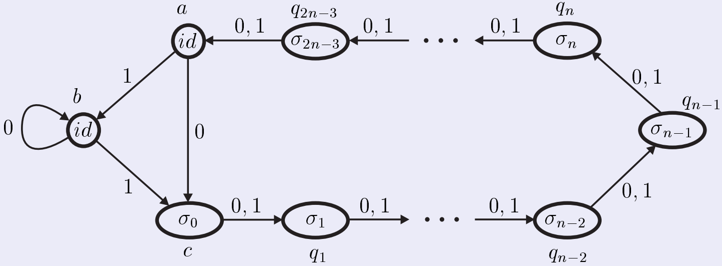


generates the free product of $2n+1$ *groups of order 2.*

Generalizations

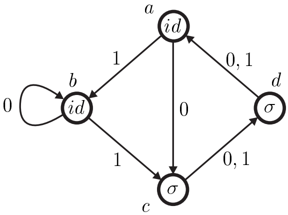
Theorem (B. Steinberg, M. Vorobets, Ya. Vorobets (2006))

The automaton

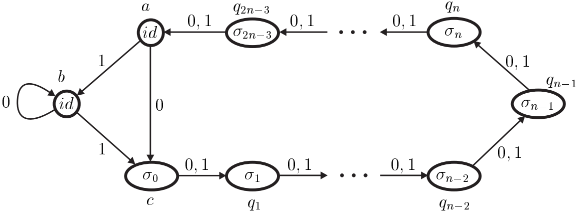


where the number of nontrivial σ_i is odd, generates the free product of $2n$ groups of order 2.

Motivating example



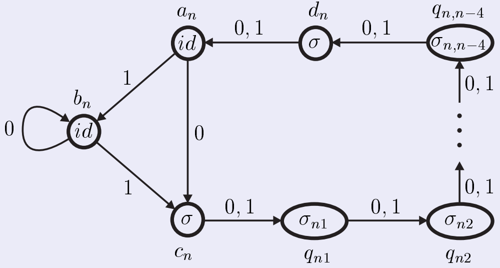
is the smallest not covered by Vorobets, Vorobets, Steinberg series



What we prove

Theorem

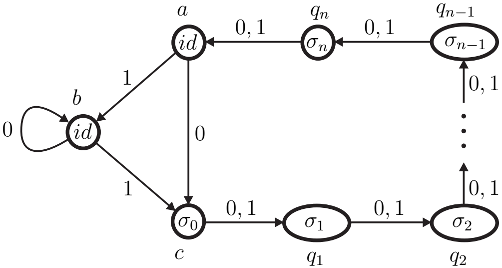
The automaton



where σ_i are chosen arbitrarily, generates the free product of n groups of order 2.

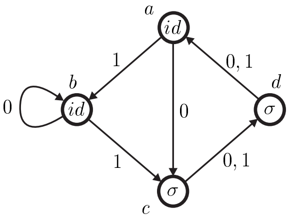
Brave conjecture

Any automaton from the family



where at least one σ_i is nontrivial generates the free product of groups of order 2

Starting Point: 4-state automaton



$$
\begin{aligned}
a &= (c, b), \\
b &= (b, c), \\
c &= (d, d)\sigma, \\
d &= (a, a)\sigma.
\end{aligned}
$$

Theorem

$G_{\mathcal{A}} \cong C_2 * C_2 * C_2 * C_2$

Dual Automata Motivation

One more way to define a self-similar group is by its action on X^*.

$a = (c, b),$	$a(0w) = 0c(w)$ $a(1w) = 1b(w)$
$b = (b, c),$	$b(0w) = 0b(w)$ $b(1w) = 1c(w)$
$c = (d, d)\sigma,$	$c(0w) = 1d(w)$ $c(1w) = 0d(w)$
$d = (a, a)\sigma,$	$d(0w) = 1a(w)$ $d(1w) = 0a(w)$

Dual Automata Motivation

It's easy to compute $gw := g(w)$ for $g \in G$ and $w \in X^*$.

$a0 \to 0c$ $a1 \to 1b$ $b0 \to 0b$ $b1 \to 1c$ $c0 \to 1d$ $c1 \to 0d$ $d0 \to 1a$ $d1 \to 0a$	$dbd001*$

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$$
\begin{array}{l||}
a0 \to 0c \\
a1 \to 1b \\
b0 \to 0b \\
b1 \to 1c \\
c0 \to 1d \\
c1 \to 0d \\
d0 \to 1a \\
d1 \to 0a
\end{array}
$$

$dbd001*$
$db1a01*$

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$a0 \to 0c$	$dbd001*$
$a1 \to 1b$	$db1a01*$
$b0 \to 0b$	$db10c1*$
$b1 \to 1c$	
$c0 \to 1d$	
$c1 \to 0d$	
$d0 \to 1a$	
$d1 \to 0a$	

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$c0 \to 1d$	$d1c00d*$
$c1 \to 0d$	$d11d0d*$
$d0 \to 1a$	$d111ad*$
$d1 \to 0a$	

Dual Automata Motivation

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$b1 \to 1c$	$db100d*$
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$d0 \to 1a$	$d111ad*$
$d1 \to 0a$	$0a11ad*$

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$c0 \to 1d$	$d1c00d*$
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$b0 \to 0b$	$db10c1*$
$b1 \to 1c$	$db100d*$
$c0 \to 1d$	$d1c00d*$
$c1 \to 0d$	$d11d0d*$
$d0 \to 1a$	$d111ad*$
$d1 \to 0a$	$0a11ad*$
	$01b1ad*$
	$011cad*$

Dual Automata Motivation

It's easy to compute $gw := g(w)$ for $g \in G$ and $w \in X^*$.

$a0 \to 0c$	$dbd001*$	Hence
$a1 \to 1b$	$db1a01*$	$d(b(d(001))) = dbd(001) = 011$
$b0 \to 0b$	$db10c1*$	and
$b1 \to 1c$	$db100d*$	$(dbd)\|_{001} = dac$
$c0 \to 1d$	$d1c00d*$	
$c1 \to 0d$	$d11d0d*$	
$d0 \to 1a$	$d111ad*$	
$d1 \to 0a$	$0a11ad*$	
	$01b1ad*$	
	$011cad*$	

Dual Automata Motivation

It's easy to compute $gw := g(w)$ for $g \in G$ and $w \in X^*$.

$$
\begin{array}{l||c}
a0 \to 0c & dbd001* \\
a1 \to 1b & db1a01* \\
b0 \to 0b & db10c1* \\
b1 \to 1c & db100d* \\
c0 \to 1d & d1c00d* \\
c1 \to 0d & d11d0d* \\
d0 \to 1a & d111ad* \\
d1 \to 0a & 0a11ad* \\
 & 01b1ad* \\
 & 011cad*
\end{array}
$$

Hence

$$d(b(d(001))) = dbd(001) = 011$$

and

$$(dbd)|_{001} = dac$$

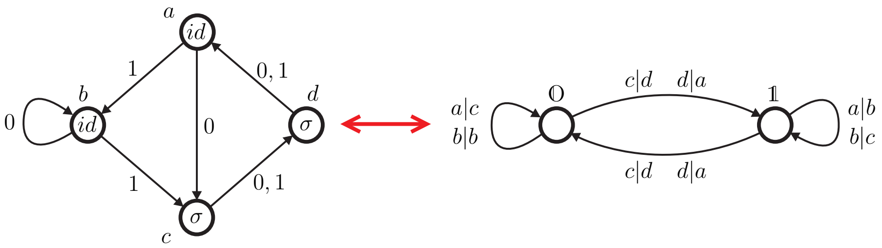
Question: Who acts on whom?

Idea of the proof

Definition

For $\mathcal{A} = (Q, X, \pi, \lambda)$ its *dual automaton* $\hat{\mathcal{A}}$ is defined by "flipping the roles" of the set of states Q and alphabet X. I.e. $\hat{\mathcal{A}} = (X, Q, \hat{\lambda}, \hat{\pi})$, where

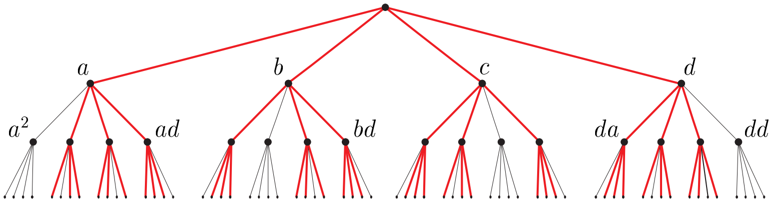
$$\hat{\lambda}(x, q) = \lambda(q, x),$$
$$\hat{\pi}(x, q) = \pi(q, x)$$



The dual group Γ is generated by dual automaton

$$\mathbb{0} = (\mathbb{0}, \mathbb{0}, \mathbb{1}, \mathbb{1})(a, c, d),$$
$$\mathbb{1} = (\mathbb{1}, \mathbb{1}, \mathbb{0}, \mathbb{0})(a, b, c, d),$$

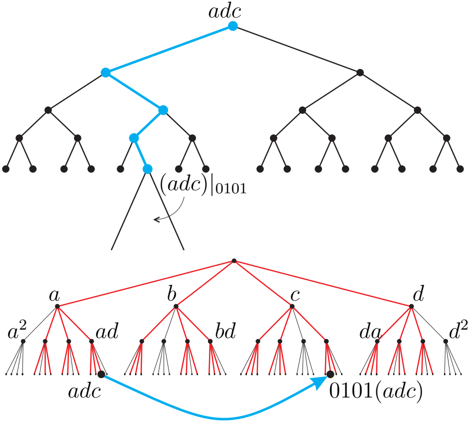
Γ acts on 4-ary tree leaving the red subtree $\hat{T}$ invariant:



Proposition

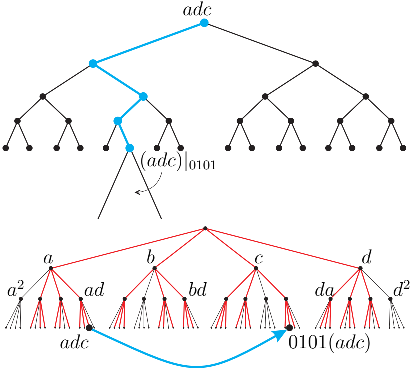
Let $G = \langle S \rangle$ be an automaton semigroup acting on X^*. And let $\hat{G}$ be its dual semigroup acting on S^*. Then for any $g \in G$ and $v \in X^*$

$$g|_v = v(g)$$



Proposition

Each level of the tree $\hat{T}$ contains at least one nontrivial element of $G_{\mathcal{A}}$. One can take $abab\cdots abc$ or $abab\cdots abac$.



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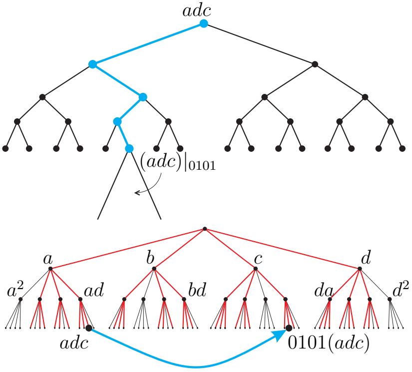


Corollary

Transitivity of Γ on $\hat{T}$ $\Rightarrow \left[G_{\mathcal{A}} \cong C_2 * C_2 * C_2 * C_2\right]$.

$$\boxed{G_{\mathcal{A}} \cong C_2 * C_2 * C_2 * C_2}$$

$\boxed{G_{\mathcal{A}} \cong C_2 * C_2 * C_2 * C_2}$ $\Longleftarrow$ $\boxed{\Gamma \text{ acts level transitively}}$

$\boxed{\Gamma \text{ is infinite}}$

$\Downarrow$

$\boxed{G_{\mathcal{A}} \cong C_2 * C_2 * C_2 * C_2}$ $\Longleftarrow$ $\boxed{\Gamma \text{ acts level transitively}}$

$\boxed{G_{\mathcal{A}} \text{ is infinite}}$ $\iff$ $\boxed{\Gamma \text{ is infinite}}$

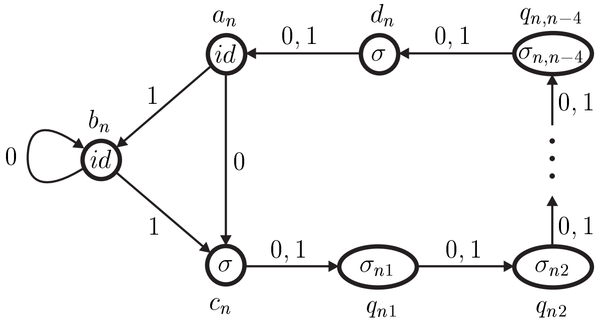
$\Downarrow$

$\boxed{G_{\mathcal{A}} \cong C_2 * C_2 * C_2 * C_2}$ $\Longleftarrow$ $\boxed{\Gamma \text{ acts level transitively}}$

$$\begin{tikzcd}
\boxed{G_{\mathcal{A}} \text{ acts level transitively}} \arrow[d, Leftrightarrow] & \\
\boxed{G_{\mathcal{A}} \text{ is infinite}} \arrow[r, Leftrightarrow] & \boxed{\Gamma \text{ is infinite}} \arrow[d, Rightarrow] \\
\boxed{G_{\mathcal{A}} \cong C_2 * C_2 * C_2 * C_2} & \boxed{\Gamma \text{ acts level transitively}} \arrow[l, Rightarrow]
\end{tikzcd}$$

$$
\begin{tikzcd}
\boxed{\begin{array}{c} G_{\mathcal{A}} \text{ acts level} \\ \text{transitively} \end{array}} & \begin{array}{l} \text{TRUE} \\ \text{(there is an algorithm)} \end{array} \arrow[l] \\
\boxed{G_{\mathcal{A}} \text{ is infinite}} \arrow[u, Leftrightarrow] & \boxed{\Gamma \text{ is infinite}} \arrow[l, Leftrightarrow] \arrow[d, Rightarrow] \\
\boxed{G_{\mathcal{A}} \cong C_2 * C_2 * C_2 * C_2} & \boxed{\begin{array}{c} \Gamma \text{ acts level} \\ \text{transitively} \end{array}} \arrow[l, Rightarrow]
\end{tikzcd}
$$

Family of automata



Theorem

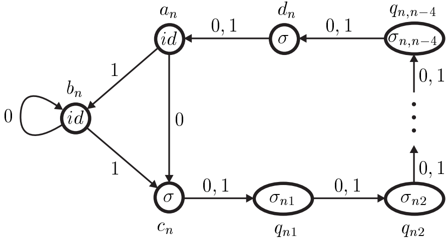
The groups $G^{(n)}$ generated by automata from the family above are isomorphic to the free products of n groups of order 2

Proposition

The dual group $\Gamma^{(n)} = \langle \mathbb{O}_n, \mathbb{1}_n \rangle$ acts on n-ary tree $T^{(n)}$:

$$\mathbb{O}_n = (\mathbb{O}_n, \mathbb{O}_n, \mathbb{1}_n, \mathbb{K}_{n1}, \ldots, \mathbb{K}_{n,n-4}, \mathbb{1}_n)(a_n\, c_n\, q_{n1} \ldots q_{n,n-4}\, d_n),$$
$$\mathbb{1}_n = (\mathbb{1}_n, \mathbb{1}_n, \mathbb{O}_n, \mathbb{L}_{n1}, \ldots, \mathbb{L}_{n,n-4}, \mathbb{O}_n)(a_n\, b_n\, c_n\, q_{n1} \ldots q_{n,n-4}\, d_n),$$

where $\mathbb{K}_{n,i} = \mathbb{O}_n$ and $\mathbb{L}_{n,i} = \mathbb{1}_n$ if $\sigma_{n,i} = id$, and $\mathbb{K}_{n,i} = \mathbb{1}_n$ and $\mathbb{L}_{n,i} = \mathbb{O}_n$ otherwise.



$$\begin{aligned}\alpha_n &= (\alpha_n, \alpha_n, \beta_n, \gamma_{n1}, \ldots, \gamma_{n,n-4}, \beta_n) \quad (a_n\, b_n)(c_n\, q_{n1} \ldots q_{n,n-4}\, d_n),\\ \beta_n &= (\beta_n, \beta_n, \alpha_n, \delta_{n1}, \ldots, \delta_{n,n-4}, \alpha_n) \quad (c_n\, q_{n1} \ldots q_{n,n-4}\, d_n),\end{aligned}$$

where $\gamma_{n,i} = \alpha_n$ and $\delta_{n,i} = \beta_n$ if $\sigma_{n,i} = id$, and $\gamma_{n,i} = \beta_n$ and $\delta_{n,i} = \alpha_n$ otherwise.

Proposition

$$\Gamma^{(n)} = \langle \alpha_n, \beta_n, \overline{(b_n\, c_n)} \rangle.$$

From the base case we know that $\Gamma^{(4)} = \Gamma$ acts transitively on $\hat{T}^{(4)}$

Lemma

For any $v \in \Gamma$ there exists $v' \in \Gamma^{(n)}$ with the following property. For any word g over $\{a_n, b_n, c_n\}$ such that $v(g)$ is also a word over $\{a_n, b_n, c_n\}$, we have $v(g) = v'(g)$.

The proof of transitivity of $\Gamma^{(n)}$ on the levels of $\hat{T}^{(n)}$ follows by induction on level.

$$
\begin{array}{ll}
g_1 g_2 g_3 \dots g_{k-1} g_k, & g_i \in \{a_n, b_n, c_n, q_1, \dots, d_n\} \\
\quad \downarrow \text{induction assumption} & \\
a_n b_n a_n \dots a_n b_n t, & t \in \{a_n, c_n, q_1, \dots, d_n\} \\
\quad \downarrow \beta_n^j & \\
a_n b_n a_n \dots a_n b_n t', & t' \in \{a_n, c_n\} \\
\quad \downarrow \text{transitivity of } \Gamma & \\
a_n b_n a_n \dots a_n b_n a_n &
\end{array}
$$