One-relator relative presentations and hyperbolicity

Anton A. Klyachko

The Beginings of The Combinatorial Group Theory

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Now, there are A LOT of results on one-relator groups... So, we consider a generalisation.

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$$\widehat{G} = \langle A \cup \{x_1, x_2, \ldots, x_n\} \mid R \cup \{w\} \rangle .$$

Here $x_1, \ldots, x_n$ are some letters (not belonging to G) and w is a word in the alphabet $G \cup \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$.

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One-relator group theory can be extended to these relative presentations if we impose some conditions on the initial group G and/or on the relator w. There are many such theorems...

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$\widehat{\mathrm{G}}$ maps onto $\left\langle \mathrm{t} \,\middle|\, \mathrm{t}^{\sum \varepsilon_\mathrm{i}} = 1 \right\rangle$. So, KC is true, unless

$$\sum \varepsilon_\mathrm{i} = 1 \quad (\textit{unimodular} \text{ case}).$$

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These results are based on Howie's diagrams and the car-crash lemma (K., 1993).

There are also multi-variable analogues of these results, but the time is limited...

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Newman Theorem (1968)

H is *hyperbolic*,

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Duncan & Howie's Theorems (1991, 1993)

If either

- G is locally indicable or
- G is involution-free and $k \geqslant 4$,

then $\widehat{G}$ is *relatively hyperbolic* with respect to G,

i.e. each word u representing 1 in H is a product of at most $C|u|$ conjugates of $w^{\pm k}$, where C does not depend on u. Here $|u|$ is the number of letters $t^{\pm 1}$ in the word $u \in G * \langle t \rangle_\infty$.

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- the word problem is solvable in A if it is solvable in B (Farb, 1998);
- the conjugacy problem is solvable in A if it is solvable in B and B is finitely generated (Bumagina, 2004);
- A possesses (or inherits from B) many other good algebraic and algorithmic properties.

Powered unimodular relators are even better

$$\widehat{G} = \left\langle G, t \;\middle|\; (g_1 t^{\varepsilon_1} \dots g_n t^{\varepsilon_n})^k = 1 \right\rangle, \quad \text{where } k \geqslant 2 \text{ and } \sum \varepsilon_i = 1.$$

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Le Thi Giang's Theorem (2009)

If G is torsion-free then $\widehat{\mathrm{G}}$ is relatively hyperbolic with respect to G.

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Giang used Howie's diagrams and a quantitative variant of the car-crash lemma (K., 1997, 2005).
We used the same approach in combination with the weight test (Gersten, 1987; Pride, 1988).

Car-crash lemma

Car-crash lemma (K., 1993)

Consider a map on a sphere. Suppose that on each face of this map there is a car that moves along the boundary of the face anticlockwise (the interior of the face remains on the left from the car) without U-turns, stops, and “infinite decelerations and accelerations”. Then there are at least two points where some cars collide.

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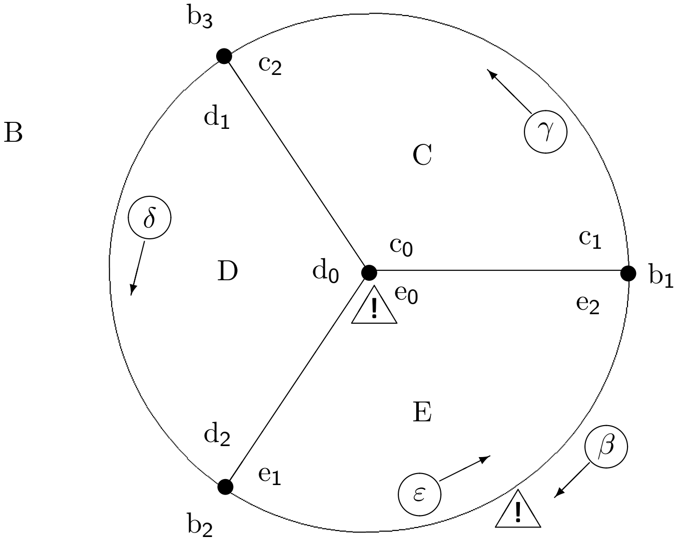
In the quantitative version of this lemma (K., 1997, 2005), there are several cars moving “regularly” around each face and the number of collisions must be large with respect to the number of faces. This allows Giang to obtain a linear isoperimetric inequality, which is equivalent to the relative hyperbolicity.

A map on a sphere



Weight test (Gersten, 1987; Pride, 1988)

Suppose we have a map on a sphere and each corner c is assigned a value (weight) $\nu(\mathrm{c})$. Then

$$\sum_{\mathrm{v}} \mathrm{K}(\mathrm{v}) + \sum_{\mathrm{D}} \mathrm{K}(\mathrm{D}) = 4.$$

Here the summations are over all vertices v and faces D of the map; and the *curvatures* $\mathrm{K}(\mathrm{v})$ and $\mathrm{K}(\mathrm{D})$ of a vertex v and a face D are defined by the formulae

$$\mathrm{K}(\mathrm{v}) \overset{\mathrm{def}}{=} 2 - \sum_{\mathrm{c}} \nu(\mathrm{c}), \qquad \mathrm{K}(\mathrm{D}) \overset{\mathrm{def}}{=} 2 - \sum_{\mathrm{c}} (1 - \nu(\mathrm{c})),$$

where the first summation is over all corners at the vertex v, and the second summation is over all corners of the face D.