Definable subsets in free and hyperbolic groups

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Abstract

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Following our work on Tarski problems, we (with Olga Kharlampovich) give a description of definable subsets in free non-abelian and torsion-free non-elementary hyperbolic groups.

First-order language of groups

The language L of group theory consists of multiplication $\cdot$, inversion $^{-1}$, and a constant symbol 1 for the identity in the group.

For a given group G one may include all elements of G as constants to the language L thus obtaining a language L_G.

If G is generated by a finite set A it suffices to include only elements of A into the language.

First-order formulas in groups

By $\phi(p_1, \ldots, p_n)$ we denote a first-order formula in the language L (or L_G) whose free variables are contained in the set $\{p_1, \ldots, p_n\}$.

We also use tuple notation for variables referring to ϕ above as to $\phi(P)$ where $P = (p_1, \ldots, p_n)$.

One may consider only first-order formulas of the type

$$\phi(P) = \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \phi_0(P, X, Y),$$

where $\phi_0(P, X, Y)$ has no quantifiers.

A formula without free variables is called a sentence.

First-order Theory

The first order theory $Th(G)$ of a group G is the set of all first-order sentences in L (or in L_G) that are true in G.

$Th(G)$ is all the information about G describable in first-order logic.

Two structures A and B (of the same type) are elementarily equivalent (symbolically $A \equiv B$) if $Th(A) = Th(B)$.

A has decidable first-order theory if $Th(A)$ is a computable set of formulas.

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Tarski

Tarski's type problems for a given class of structures $\mathcal{C}$:

- When $A \equiv B$ for $A, B \in \mathcal{C}$.
- Describe when $Th(A)$ is decidable for $A \in \mathcal{C}$.

Solution to Tarski type problems for free groups:

Theorem [Kharlampovich-Myasnikov, Sela]

$Th(F_n) = Th(F_m), m, n > 1.$

Theorem [Kharlampovich and Myasnikov]

The elementary theory $Th(F)$ of a free group F even with constants from F in the language is decidable.

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Malcev's problems

Malcev: Let F be a free non-abelian group.

1) Describe definablle sets in F;
2) Describe definable subgroups in F;
3) Is the commutant $[F, F]$ of F definable in F?

A subset $S \subseteq G^n$ is definable in a group G if there exists a first-order formula $\phi(P)$ in L_G such that S is precisely the set of tuples in G^n where $\phi(P)$ holds:

$$S = \{g \in G^n \mid G \models \phi(g)\}$$

Sometimes we say that S is definable without parameters if ϕ does not involve constants from G.

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Examples of definable sets in a group G

Algebraic sets: let $W(P, A) = 1$ be an equation (with constants) in a group G. Then the algebraic set

$$V_G(W) = \{g \in G^n \mid W(g, A) = 1\}$$

is definable in G.

Examples of definable sets in a group G

Verbal sets: let $w(x_1, \dots, x_n) \in F(X)$ be a group word. Then the set

$$w[G] = \{g \in G \mid g = w(h_1, \dots, h_n) \text{ for some } h_1, \dots, h_n \in G\}$$

is a verbal subset of G defined by w. It is defined in G by the formula

$$\phi(p) = \exists y_1 \dots \exists y_n (p = w(y_1, \dots, y_n)).$$

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Examples of definable sets

Bases in F_2: the set of all bases in $F_2 = F_2(a, b)$ is definable.

This is based on Nielsen's Theorem: elements $g, h \in F_2$ form a basis iff $[g, h]$ is conjugated either to $[a, b]$ or $[b, a]$.

Hence the set of bases in F_2 is defined by the following formula

$$\phi(p_1, p_2) = \exists z([p_1, p_2] = z^{-1}[a, b]z \ \vee \ [p_1, p_2] = z^{-1}[b, a]z).$$

Primitive elements in F_2: the set of all primitive elements in $F_2 = F_2(a, b)$ is defined by the following formula

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Examples of definable subgroups

The following subgroups are definable in any group G:

1) The center $Z(G) = \{g \in G \mid \forall x [g, x] = 1\}$.
2) The centralizer of a finite subset $M = \{g_1, \ldots, g_m\} \subseteq G$:

$$C_G(M) = \{g \in G \mid \wedge_{i=1}^{m} [g, g_i] = 1\};$$

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The verbal subgroup $w(G)$ has finite width if there is a number k such that every element in $w(G)$ is a product of at most k values of the word w in G or their inverses.

A verbal subgroup $w(G)$ of finite width is definable in G (without parameters):

$$w(G) = \{g \in G \mid \exists X_1 \ldots \exists X_k (g = w(X_1)^{\pm 1} \ldots w(X_k)^{\pm 1}\}.$$

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Why subgroups or the commutant?

Why Malcev asked about definability of subgroups, in particular about the commutant $[G, G]$ of G?

Elementary equivalent free abelian groups

Let A_m be a free abelian group of rank m.

The verbal subgroup A_m^2 has width 1 in A_m hence definable.

A_m/A_m^2 is a vector space of dimension m over the field $\mathbb{Z}_2$ of two elements.

Using definability of A_m^2 one can write a sentence D_m (without parameters) which states that the dimension of the space A_m/A_m^2 is precisely m.

Hence
Theorem: two free abelian groups of finite rank are elementarily equivalent iff they are isomorphic.

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Elementary equivalent free nilpotent groups

Let G be a finitely generated free nilpotent group of rank m and class c.

Fact: The commutant $[G, G]$ has finite width.

Hence definable in G. So the abelianization $G/[G, G] \simeq A_m$ is interpretable in G. Again, one can write down a sentence stating that the rank of the abelianization of G is precisely m.

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Elementary equivalent free solvable groups groups

Let G be a free solvable group of rank m and class c, so

$$G = G^{(1)} > G^{(2)} = [G, G] > \ldots > G^{(c)} = [G^{(c-1)}, G^{(c-1)}] > 1$$

The last non-trivial commutant $G^{(c)}$ is abelian and $G^{(c)} = C_G(g)$ for any $1 \neq g \in G^{(c)}$. Hence definable. By induction $[G, G]$ is definable in G. Hence one can describe the rank of the abelianization of G by sentences.

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Verbal subgroups in F

Theorem [Rhemtulla]: Proper verbal subgroups in non-abelian free group F are of infinite width.

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Definable sets in F

Quantifier elimination [Sela, 2006]

Every formula in the theory of F is equivalent to a boolean combination of $\exists\forall$-formulas.

Every definable set over F is defined by some boolean combination of formulas

$$\phi(P) = \exists X \forall Y (\vee_{i=1}^{k} (U_i(P, X, Y) = 1 \wedge V_i(P, X, Y) \neq 1)),$$

where X, Y, P are tuples of variables.

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Effective Quantifier Elimination [KM, 2012]

Every formula in the theory of F is equivalent to some boolean combination of formulas

$$\exists X \forall Y (U(P,X) = 1 \wedge V(P,X,Y) \neq 1),$$

where X, Y, P are tuples of variables.
Moreover, there is an algorithm that for a given formula $\phi(P)$ finds an equivalent boolean combination of the type above.

Observe, that unlike arbitrary $\exists\forall$-formulas the "basic" $\exists\forall$-formulas

$$\psi(P) = \exists X \forall Y (U(P,X) = 1 \wedge V(P,X,Y) \neq 1)$$

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Effective description of definable sets

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There is an algorithm that for a given definable set $S \subseteq F^n$ finds some decomposition of S as a boolean combination of basic $\exists\forall$-subsets, i.e, subsets defined by formulas of the type

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Pieces

Definition

A *piece* of a word $u \in F$ is a non-trivial subword v that appears in u in two different ways (maybe the second time as v^{-1}).

Parametrization in F

Definition

A proper subset P of F admits parametrization if it is a set of all words p that satisfy a given system of equations (with coefficients) without cancellations in the form

$$p \circeq w_t(y_1, \ldots, y_n), t = 1, \ldots, k,$$

where for all $i = 1, \ldots, n$, $y_i \neq 1$, and each y_i appears at least twice in the system and each variable y_i that appears in w_1 is a piece of p.

Definition

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Definition

A finite union of subsets of F admitting parametrization is called a multipattern in F.

Wicks' forms [Wicks 1962]

An element g in a free group F is a commutator if and only if the cyclically reduced form of g up to a cyclic permutation can be written without cancellation in the form

$$abca^{-1}b^{-1}c^{-1}$$

for some $a, b, c \in F$.

This gives a nice parametrization of the set of commutators in F as a multipattern.

Parametrization of commutators in F

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Wicks' forms

Corollary

g is a commutator in F iff g decomposes without cancellation in one of the following forms with all factors non-trivial:

- $g = abca^{-1}b^{-1}c^{-1}$,
- $g = aba^{-1}b^{-1}$ (when precisely one of a, b, c above is trivial),
- $g = 1$ (when more than one of a, b, c above is trivial),
- $g = a_2bca_2^{-1}a_1^{-1}b^{-1}c^{-1}a_1$ (cyclically permuting a_1 for $a = a_1a_2$ in the first form),
- $g = a_2ba_2^{-1}a_1^{-1}b^{-1}a_1$ (cyclically permuting a_1 for $a = a_1a_2$ in the second form),
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- $g = a_2ba_2^{-1}a_1^{-1}b^{-1}a_1$ (cyclically permuting a_1 for $a = a_1a_2$ in the second form),
- $g = wqw^{-1}$, where q is one of the non-trivial forms above.

Wicks' forms

Corollary

g is a commutator in F iff g decomposes without cancellation in one of the following forms with all factors non-trivial:

- $g = abca^{-1}b^{-1}c^{-1}$,
- $g = aba^{-1}b^{-1}$ (when precisely one of a, b, c above is trivial),
- $g = 1$ (when more than one of a, b, c above is trivial),
- $g = a_2bca_2^{-1}a_1^{-1}b^{-1}c^{-1}a_1$ (cyclically permuting a_1 for $a = a_1a_2$ in the first form),
- $g = a_2ba_2^{-1}a_1^{-1}b^{-1}a_1$ (cyclically permuting a_1 for $a = a_1a_2$ in the second form),
- $g = wqw^{-1}$, where q is one of the non-trivial forms above.

Parametrization in F^m

Definition

A *piece* of a tuple of reduced words $(u_1, \ldots, u_m)$, $u_j \in F$ is a non-trivial subword v that appears in two different ways as a subword in $u_1, \ldots, u_m$.

Definition

A proper subset P of F^m admits parametrization if (up to a suitable permutation of components) $P = F^k \times S$ where S is a set of all tuples $p = (p_1, \ldots, p_{m-k}) \in F^{m-k}$ satisfying in F the following system $W = 1$ of equations (with coefficients) without cancellation:

$$p_j \mathrel{\hat{=}} w_{1j}(y_1, \ldots, y_n), \ldots, p_j \mathrel{\hat{=}} w_{k_j,j}(y_1, \ldots, y_n), \qquad (1)$$

where $j = 1, \ldots, m-k$ and for each y_i the following holds:

- $y_i \neq 1$
- y_i occurs at least twice in the system,
- each y_i that occurs in some w_{1j} is a piece of the tuple p.

Multipatterns in F^m

Definition

A subset $S \subseteq F^n$ is called a multipattern if S is a finite union $S = S_1 \cup \ldots \cup S_m$ of subsets S_i each of which admits a parametrization in F^n.

If a system $W_i = 1$ is a parametrization of S_i then the tuple $W = (W_1, \ldots, W_m)$ is a description of the multipattern S.

A subset of a multipattern is a sub-multipattern.

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Diophantine subsets of F^n

We consider first the Diophantine subsets of F^n, i.e., sets which are defined by formulas of the type

$$\psi(P) = \exists Y(U(Y, P) = 1),$$

Theorem [KM, 2012]

Every proper subset S of F^n defined by a formula

$$\psi(P) = \exists Y(U(Y, P) = 1)$$

is a multipattern. Furthermore, there exists an algorithm that for a given Diophantine set S defined by a formula $\psi(P)$ as above finds a description $W = (W_1, \ldots, W_m)$ of S as a multipattern.

Basic $\exists\forall$-subsets of F^n

Theorem [KM 2012]

Let S be a basic $\exists\forall$-subset of F^n defined by a formula

$$\psi(P) = \exists X \forall Y (U(P, X) = 1 \wedge V(P, X, Y) \neq 1).$$

Then either S or its complement $\neg S$ is a sub-multipattern in F^n.

Furthermore, there exists an algorithm that for a given basic $\exists\forall$-subset S of F^n defined by a formula $\psi(P)$ as above finds a description $W = (W_1, \ldots, W_m)$ of a multipattern that contains S or $\neg S$ (whichever case holds).

Definable subsets of F^n

Theorem [KM 2012]

Let S be a definable in F subset of F^n. Then either S or its complement $\neg S$ is a sub-multipattern.

Furthermore, there exists an algorithm that for a given definable subset S of F^n defined by a formula $\psi(P)$ finds a description $W = (W_1, \ldots, W_m)$ of a multipattern that contains S or $\neg S$ (whichever case holds).

This follows from the Effective Quantifier Elimination and the result on basic $\exists\forall$-subset of F^n above. Roughly speaking, if S is a boolean combination of sub-multipatterns in F^n then either S or $\neg S$ is a sub-multipattern in F^n.

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Compare to other known result

It is interesting to compare the results above with other known results on definable sets in algebraic structures:

- algebraically closed fields,
- real closed fields,
- abelian groups (or modules).

Algebraically closed fields

Tarski

Let K be an algebraically closed field. Then definable subsets of K are precisely finite or co-finite ones.

So definable subsets of K are either very small or very large.

Definition

Recall that a subset $T \subseteq F = F(X)$ is generic if

$$\rho_n(T) = \frac{|T \cap B_n(X)|}{|B_n|} \to 1, \ \textit{if} \ n \to \infty,$$

where $B_n(X)$ is the ball of radius n in the Cayley graph of $F(X)$.

A set is negligible in F if its complement is generic.

Generic and negligible sets

Definition

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Multipatterns are small

Theorem

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So definable subsets of F are either negligible or generic.

Definition (Bestvina-Feighn)

A subset P of F is *negligible* if there exists $\epsilon > 0$ such that all but finitely many $p \in P$ have a piece such that

$$\frac{length(piece)}{length(p)} \geq \epsilon.$$

A complement of a negligible subset is *co-negligible*.

I will refer to such sets as BF-negligible, or BF-co-negligible.

BF

Let S be a subset of F.

If S contains a coset of a non-abelian subgroup of F then S is not BF-negligible.

In particular, a proper non-abelian subgroup of F is neither BF-negligible nor BF-co-negligible.

Proof. If G is a non-abelian subgroup of F and $x, y \in G$ such that $[x, y] \neq 1$, then an infinite set

$$\{xyxy^2x \dots xy^ix \mid \ i \in \mathbb{N}\}$$

is not BF-negligible, as well as its shift

$$\{fxyxy^2x \dots xy^ix \mid \ i \in \mathbb{N}\}$$

by a fixed element $f \in F$.

BF

If F is a free group of rank > 2 then the set of primitive elements of F is neither BF-negligible nor BF-co-negligible.

Proof. Let a, b, c be three distinct elements in the basis of F and denote $F_2 = F(a, b)$ The set of all primitive elements in F contains cF_2, and its complement contains $< [a, b], c^{-1}[a, b]c >$.

Properties of BF-negligible sets

KM

A multipattern, as well as a sub-multipattern, in F is BF-negligible.

Indeed, suppose S admits a parametrization $W = 1$. Let m be the length of the word w_1 (as a word in variables y_i's and constants). Then the set S is negligible with $\epsilon = 1/m$.

Definable sets and BF-negligibility

Bestvina and Feighn: claimed (not published yet) that every definable set of F is either negligible or co-negligible.

Now the claim follows from our description of definable subsets S of F.

Indeed, if $S \subseteq F$ is definable in F then either S or $\neg S$ is a sub-multipattern, hence BF-negligible.

Definable sets and BF-negligibility

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Indeed, if $S \subseteq F$ is definable in F then either S or $\neg S$ is a sub-multipattern, hence BF-negligible.

Solution to Malcev's problem on subgroups

Theorem [BF, KM]

Proper non-abelian subgroups of F are not definable.

Let G be a group generated by a set A, $\nu : F(A) \to G$ the canonical projection.

Definition

A proper subset P of G admits parametrization if P is the image under ν of a set $\tilde{P}$ in $F(A)$ that admits parametrization in $F(A)$ and there exist constants λ, c and D such that for each $p \in P$ there is a pre-image $\tilde{p} \in \tilde{P}$ such that the path corresponding to $\tilde{p}$ in the Cayley graph of G is (λ, c)-quasigeodesic in D-neighborhood of the geodesic path for p.

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Theorem [KM, 2012]

Let G be a non-elementary torsion free hyperbolic group. Then for every definable subset P of G, either P or its complement $\neg P$ is a sub-multipattern in G.

The same holds for definable subsets of G^n.

Thanks!