Actions, length functions, and non-Archimedean words

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This talk is based on joint results with A. Myasnikov and D. Serbin.

The starting point

Theorem . A group G is free if and only if it acts freely on a tree.

Free action $=$ no inversion of edges and stabilizers of vertices are trivial.

Ordered abelian groups

$\Lambda =$ an ordered abelian group (any $a, b \in \Lambda$ are comparable and for any $c \in \Lambda$: $a \leq b \Rightarrow a + c \leq b + c$).

Examples:

Archimedean case:

$\Lambda = \mathbb{R}$, $\Lambda = \mathbb{Z}$ with the usual order.

Non-Archimedean case:

$\Lambda = \mathbb{Z}^2$ with the right lexicographic order:

$$(a, b) < (c, d) \iff b < d \text{ or } b = d \text{ and } a < c.$$

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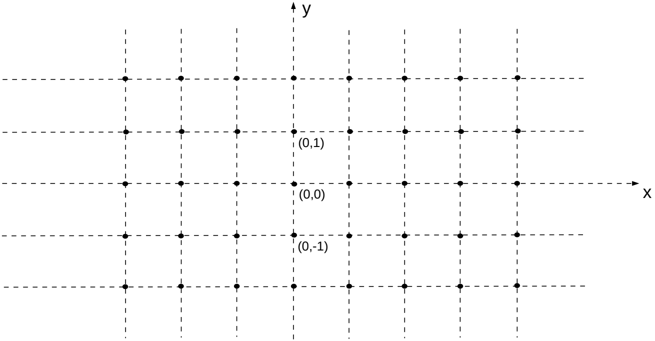
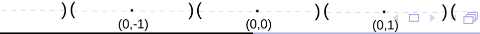
$\mathbb{Z}^2$ with the right-lex ordering



One-dimensional picture



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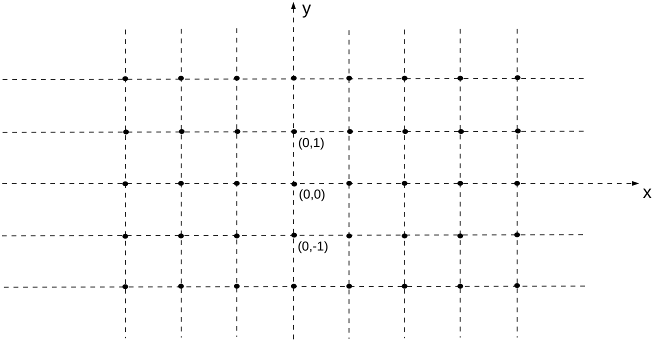

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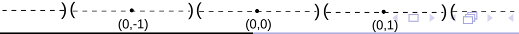

Λ-trees

Morgan and Shalen (1985) defined Λ-trees:
A Λ-tree is a metric space (X, p) (where $p : X \times X \to \Lambda$) which satisfies the following properties:

1) (X, p) is geodesic,
2) if two segments of (X, p) intersect in a single point, which is an endpoint of both, then their union is a segment,
3) the intersection of two segments with a common endpoint is also a segment.

Alperin and Bass (1987) developed the theory of Λ-trees and stated the fundamental research goals:

Find the group theoretic information carried by an action on a Λ-tree.

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> Find the group theoretic information carried by an action on a Λ-tree.

Generalize Bass-Serre theory (for actions on $\mathbb{Z}$-trees) to actions on arbitrary Λ-trees.

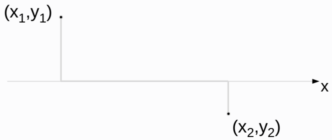
Examples for $\Lambda = \mathbb{R}$

$X = \mathbb{R}$ with usual metric.

A geometric realization of a simplicial tree.

$X = \mathbb{R}^2$ with metric d defined by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1| + |y_2| + |x_1 - x_2| & \text{if } x_1 \neq x_2 \\ |y_1 - y_2| & \text{if } x_1 = x_2 \end{cases}$$



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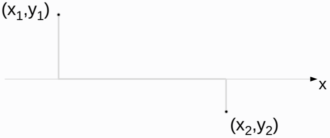

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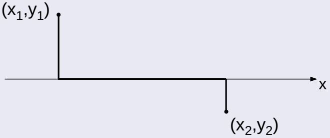

Finitely generated $\mathbb{R}$-free groups

Rips' Theorem [Rips, 1991 - not published]

A f.g. group acts freely on $\mathbb{R}$-tree if and only if it is a free product of surface groups (except for the non-orientable surfaces of genus 1,2, 3) and free abelian groups of finite rank.

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Properties

Some properties of groups acting freely on Λ-trees (Λ-free groups)

1. The class of Λ-free groups is closed under taking subgroups and free products.
2. Λ-free groups are torsion-free.
3. Λ-free groups have the CSA-property (maximal abelian subgroups are malnormal).
4. Commutativity is a transitive relation on the set of non-trivial elements.
5. Any two-generator subgroup of a Λ-free group is either free or free abelian.

The Fundamental Problem

The following is a principal step in the Alperin-Bass' program:

Open Problem [Rips, Bass]

Describe finitely generated groups acting freely on Λ-trees.

Here "describe" means "describe in the standard group-theoretic terms".

We solved this problem for finitely presented groups.
Λ-free groups = groups acting freely on Λ-trees.

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Non-Archimedean actions

Theorem (H.Bass, 1991)

A finitely generated $(\Lambda \oplus \mathbb{Z})$-free group is the fundamental group of a finite graph of groups with properties:

- vertex groups are Λ-free,
- edge groups are maximal abelian (in the vertex groups),
- edge groups embed into Λ.

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$\mathbb{Z}^n$-free groups

Theorem [Kharlampovich, Miasnikov, Remeslennikov, 96]

Finitely generated fully residually free groups are $\mathbb{Z}^n$-free.

Examples of $\mathbb{Z}^n$-free groups:

$\mathbb{R}$-free groups,

$\langle x_1, x_2, x_3 \mid x_1^2 x_2^2 x_3^2 = 1 \rangle$ is $\mathbb{Z}^2$-free (but is neither $\mathbb{R}$-free, nor fully residually free).

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Actions on $\mathbb{R}^n$-trees

Theorem [Guirardel, 2003]

A f.g. freely indecomposable $\mathbb{R}^n$-free group is isomorphic to the fundamental group of a finite graph of groups, where each vertex group is f.g. $\mathbb{R}^{n-1}$-free, and each edge group is cyclic.

However, the converse is not true.

Corollary A f.g. $\mathbb{R}^n$-free group is hyperbolic relative to abelian subgroups.

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From actions to length functions

Let G be a group acting on a Λ-tree (X, d). Fix a point $x_0 \in X$ and consider a function $l : G \to \Lambda$ defined by

$$l(g) = d(x_0\ ,\ gx_0)$$

l is called a based length function on G with respect to x_0, or a Lyndon length function.

l is free if the underlying action is free.

Example. In a free group F, the function $f \to |f|$ is a free $\mathbb{Z}$-valued (Lyndon) length function.

Regular action

Definition

Let G act on a Λ-tree Γ. The action is regular with respect to $x \in \Gamma$ if for any $g, h \in G$ there exists $f \in G$ such that $[x, fx] = [x, gx] \cap [x, hx]$.

Comments

Let G act on a Λ-tree (Γ, d). Then the action of G is regular with respect to $x \in \Gamma$ if and only if the length function $l_x : G \to \Lambda$ based at x is regular.
Let G act minimally on a Λ-tree Γ. If the action of G is regular with respect to $x \in \Gamma$ then all branch points of Γ are G-equivalent.
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F.g. $\mathbb{Z}^n$-free groups

Theorem

[KMRS] A finitely generated group G is complete $\mathbb{Z}^n$-free if and only if it can be obtained from free groups by finitely many length-preserving separated HNN extensions and centralizer extensions.

Theorem

[KMRS] Every finitely generated $\mathbb{Z}^n$-free group G has a length-preserving embedding into a finitely generated complete $\mathbb{Z}^n$-free group H. Moreover, such an embedding can be found algorithmically.

F.g. $\mathbb{Z}^n$-free groups

Theorem

[KMRS] A finitely generated group G is $\mathbb{Z}^n$-free if and only if it can be obtained from free groups by a finite sequence of length-preserving amalgams, length-preserving separated HNN extensions, and centralizer extensions.

Main Theorems

Theorem

[KMS] Any f.p. group G with a regular free length function in an ordered abelian group Λ can be represented as a union of a finite series of groups

$$G_1 < G_2 < \cdots < G_n = G,$$

where

1. G_1 is a free group,
2. G_{i+1} is obtained from G_i by finitely many HNN-extensions in which associated subgroups are maximal abelian, finitely generated, and length isomorphic as subgroups of Λ.

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[KMS] Any finitely presented Λ-free groups is $\mathbb{R}^n$-free.

Theorem

[KMS] Any finitely presented group Λ-free group $\tilde{G}$ can be isometrically embedded into a finitely presented group G that has a free regular length function in Λ. Moreover G has a free regular length function in $\mathbb{R}^n$ ordered lexicographically for an appropriate $n \in \mathbb{N}$.

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Main Theorems

Theorem

Any finitely presented Λ-free group G can be obtained from free groups by a finite sequence of amalgamated free products and HNN extensions along maximal abelian subgroups, which are free abelain groups of finite rank.

Chiswell, 2001: If G is a finitely generated Λ-free group, is G Λ_0-free for some finitely generated abelian ordered group Λ_0?

Theorem

Let G be a finitely presented group with a free Lyndon length function $l : G \to \Lambda$. Then the subgroup Λ_0 generated by $l(G)$ in Λ is finitely generated.

Main Theorems

The following result concerns with abelian subgroups of Λ-free groups. For $\Lambda = \mathbb{Z}^n$ it follows from [KMRS, 2008], for $\Lambda = \mathbb{R}^n$ it was proved by Guirardel. The statement 1) below answers the question of Chiswell in the affirmative for finitely presented Λ-free groups.

Theorem

Let G be a finitely presented Λ-free group. Then:

1) every abelian subgroup of G is a free abelian group of finite rank, which is uniformly bounded from above by the rank of the abelianization of G.
2) G has only finitely many conjugacy classes of maximal non-cyclic abelian subgroups,
3) G has a finite classifying space and the cohomological dimension of G is at most $\max\{2, r\}$ where r is the maximal

Main Theorems

Corollary

Every finitely presented Λ-free group is hyperbolic relative to its non-cyclic abelian subgroups.

Main Theorems

The following results answers affirmatively in the strongest form to the Problem (GO3) from the Magnus list of open problems in the case of finitely presented groups.

Corollary

Every finitely presented Λ-free group is biautomatic.

Theorem

Every finitely presented Λ-free group G has a quasi-convex hierarchy.

Main Theorems

Theorem

Every finitely presented Λ-free group is locally quasi-convex.

Main Theorems

Since a finitely generated $\mathbb{R}^n$-free group G is hyperbolic relative to to its non-cyclic abelian subgroups and G admits a quasi-convex hierarchy then recent results of D. Wise imply the following.

Corollary

Every finitely presented Λ-free group G is virtually special, that is, some subgroup of finite index in G embeds into a right-angled Artin group.

Chiswell, 2001: Is every Λ-free group orderable, or at least right-orderable?

Theorem

Every finitely presented Λ-free group is right orderable.

Main Theorems

Theorem

Every finitely presented Λ-free group is linear and, therefore, residually finite and equationally noetherian.

Main Theorems

The structural results of the previous section give solution to many algorithmic problems on finitely presented Λ-free groups.

Theorem

Let G be a finitely presented Λ-free group. Then the following algorithmic problems are decidable in G:

- the Word Problem;
- the Conjugacy Problems;
- the Diophantine Problem (decidability of arbitrary equations in G).

Main Theorems

Theorem ofGuirardel combined with results of F. Dahmani and D. Groves implies the following two corollaries.

Corollary

Let G be a finitely presented Λ-free group. Then:

- G has a non-trivial abelian splitting and one can find such a splitting effectively,
- G has a non-trivial abelian JSJ-decomposition and one can find such a decomposition effectively.

Main Theorems

Corollary

The Isomorphism Problem is decidable in the class of finitely presented groups that act freely on some Λ-tree.

Corollary

The Subgroup Membership Problem is decidable in every finitely presented Λ-free group.

Infinite words

Let Λ be a discretely ordered abelian group with a minimal positive element 1_Λ and $X = \{x_i \mid i \in I\}$ be a set.

An **Λ-word** is a function

$$w : [1_\Lambda, \alpha] \to X^\pm, \quad \alpha \in \Lambda.$$

$|w| = \alpha$ is called the length of w.

w is **reduced** $\iff$ no subwords xx^{-1}, $x^{-1}x$ $(x \in X)$.

$R(\Lambda, X) =$ the set of all reduced Λ-words.

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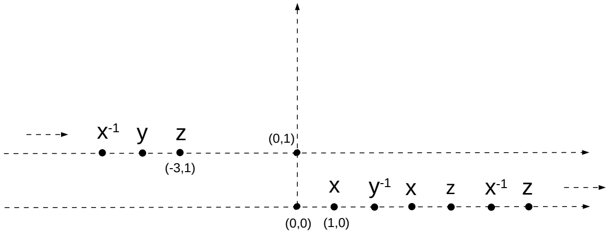
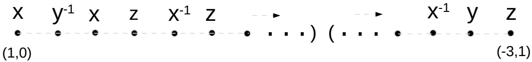
Example.

Let $X = \{x, y, z\}$, $\Lambda = \mathbb{Z}^2$



In "linear" notation



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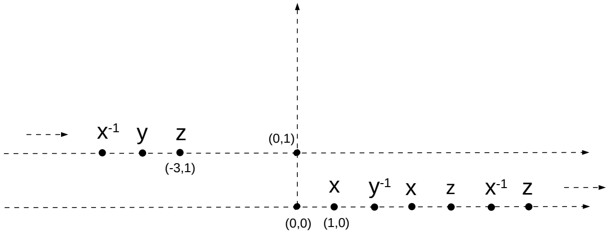

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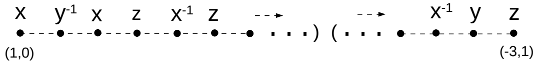
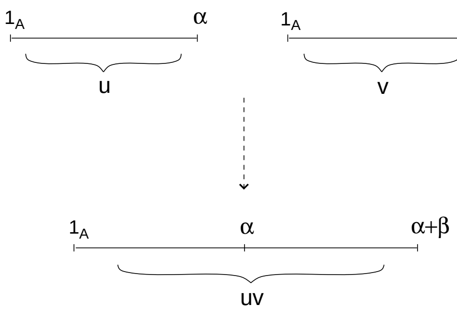
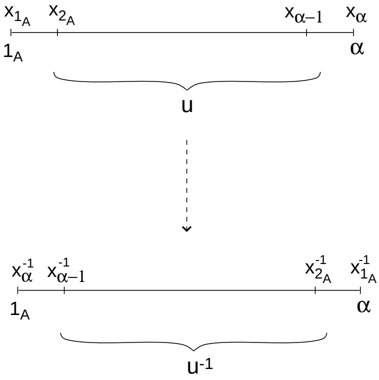


Concatenation of Λ-words:

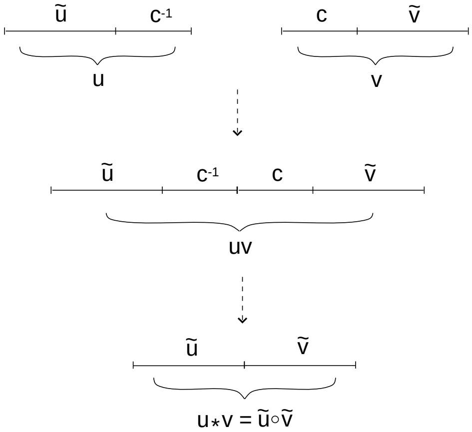


We write $u \circ v$ instead of uv in the case when uv is reduced.

Inversion of Λ-words:



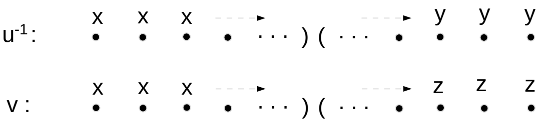
Multiplication of Λ-words:



The partial group $R(\Lambda, X)$

The multiplication on $R(\Lambda, X)$ is partial, it is not everywhere defined!

Example. $u, v \in R(\mathbb{Z}^2, X)$



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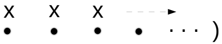

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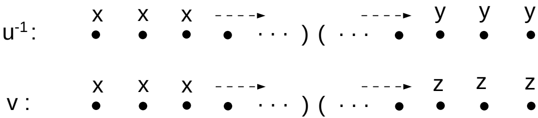

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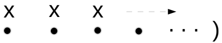

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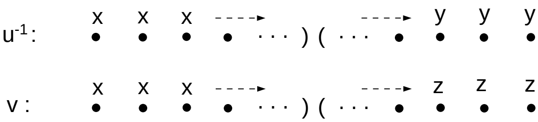
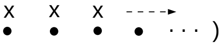


Hence, the common initial part of u^{-1} and v is



Cyclic decompositions

$v \in R(\Lambda, X)$ is **cyclically reduced** if $v(1_A)^{-1} \neq v(|v|)$.

$v \in R(\Lambda, X)$ admits a **cyclic decomposition** if

$$v = c^{-1} \circ u \circ c,$$

where $c, u \in R(A, \Lambda)$ and u is cyclically reduced.

Denote by $CDR(A, \Lambda)$ the set of all words from $R(\Lambda, X)$ which admit a cyclic decomposition.

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From Non-Archimedean words - to length functions

Theorem [Myasnikov-Remeslennikov-Serbin, 2003]

Let Λ be a discretely ordered abelian group and X a set. If G is a subgroup of $CDR(\Lambda, X)$ then the function $L_G : G \to \Lambda$, defined by $L_G(g) = |g|$, is a free Lyndon length function.

Corollary.

To show that a group G acts on a Λ-tree - embed G into $CDR(\Lambda, X)$.

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From Length functions - to Non-Archimedean words

Theorem [Chiswell], 2004

Let Λ be a discretely ordered abelian group. If $L : G \to \Lambda$ is a free Lyndon length function on a group G then there exists an embedding $\phi : G \to CDR(\Lambda, X)$ such that $|\phi(g)| = L(g)$ for every $g \in G$.

Corollary. Let Λ be an arbitrary ordered abelian group. If $L : G \to \Lambda$ is a free Lyndon length function on a group G then there exists a length preserving embedding $\phi : G \to CDR(\Lambda', X)$, where $\Lambda' = \Lambda \oplus \mathbb{Z}$ with the lex order.

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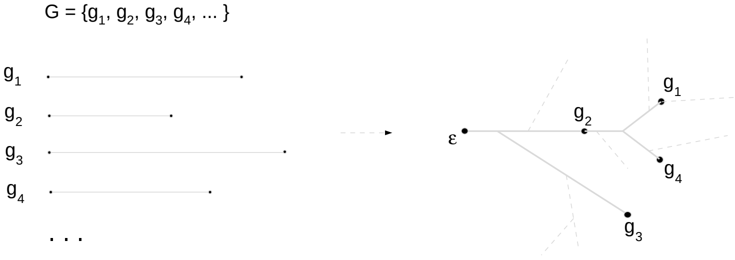
From Non-Archimedean words - to free actions

Infinite words $\Longrightarrow$ Length functions $\Longrightarrow$ Free actions

Shortcut

If $G \hookrightarrow CDR(\Lambda, X)$ then G acts by isometries on the canonical Λ-tree $\Gamma(G)$ labeled by letters from $X^{\pm}$.



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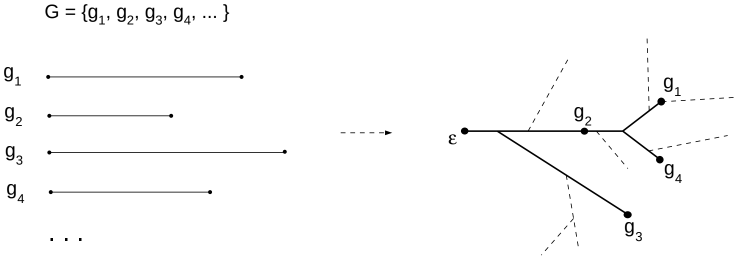

Regular free actions

A length function $l : G \to A$ is called *regular* if it satisfies the *regularity* axiom:

(L6) $\forall\, g, f \in G,\ \exists\ u, g_1, f_1 \in G$:

$$g = u \circ g_1 \ \&\ f = u \circ f_1 \ \&\ l(u) = c(g, f).$$

Complete subgroups

Let $G \leq CDR(\Lambda, X)$ be a group of infinite words.

Complete subgroups

$G \leq CDR(\Lambda, X)$ is complete if G contains the common initial segment $c(g, h)$ for every pair of elements $g, h \in G$.

Regular length functions

A Lyndon length function $L : G \to \Lambda$ is regular if there exists a length preserving embedding $G \to CDR(\Lambda, X)$ onto a complete subgroup.

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Complete subgroups

Example. Let $F(x,y)$ be a free group and $H = \langle x^2y^2,\ xy\rangle$ be its subgroup.

F has natural free $\mathbb{Z}$-valued length function $l_F : f \to |f|$. Hence, l_F induces a length function l_H on H.

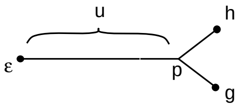
l_F is regular, but l_H is not

Take $g = xy^{-1}x^{-2},\ h = xy^{-1}x^{-1}y$ in F. Then

$$g, h \in H, \quad \text{but} \quad com(g,h) = xy^{-1}x^{-1} \notin H.$$

Branch points and completeness

A vertex $p \in \Gamma(G)$ is a branch point if it is the terminal endpoint of the common initial segment $u = com(g, h)$ of $g, h \in G$.



Completeness $\Longrightarrow$ all branch points are in one G-orbit of Γ

Conjecture

Every finitely generated complete Λ-free group is finitely presented.

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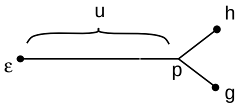

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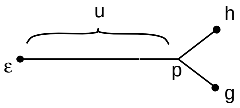

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Elimination process

Elimination Process (EP) is a dynamical (rewriting) process of a certain type that transforms formal systems of equations in groups or semigroups (or band complexes, or foliated 2-complexes, or partial isometries of multi-intervals) .
Makanin (1982): Initial version of EP.
Makanin's EP gives a decision algorithm to verify consistency of a given system of equations - decidability of the Diophantine problem over free groups.

Razborov's process

Razborov (1987): developed EP much further.
Razborov's EP produces **all solutions** of a given system in F.
The *coordinate group* of $S = 1$: $F_{R(S)} = F(A \cup X)/Rad(S)$

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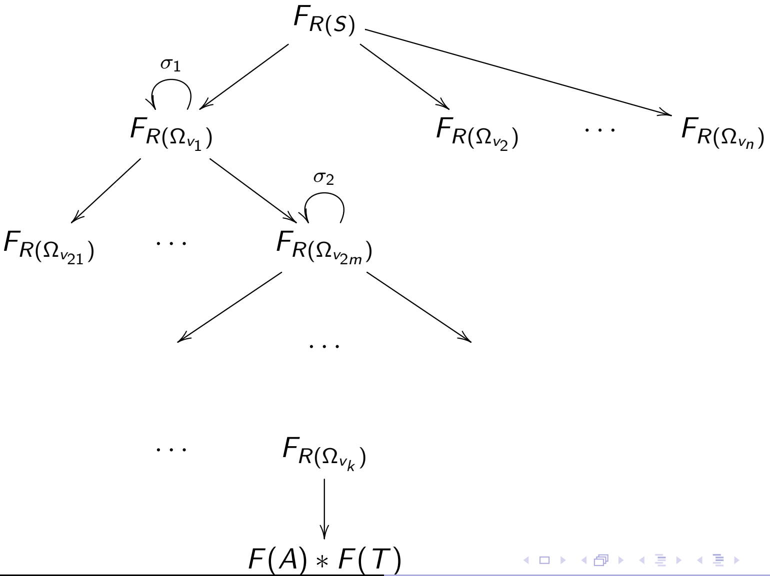
$F_{R(S)}$
σ_1
$F_{R(\Omega_{v_1})}$
$F_{R(\Omega_{v_2})}$
$\cdots$
$F_{R(\Omega_{v_n})}$
σ_2
$F_{R(\Omega_{v_{21}})}$
$\cdots$
$F_{R(\Omega_{v_{2m}})}$
$\cdots$
$\cdots$
$F_{R(\Omega_{v_k})}$
$F(A) * F(T)$

Kharlampovich - Myasnikov (1998):

Refined Razborov's process.
Effective description of solutions of equations in free (and fully residually free) groups in terms of very particular **triangular systems** of equations.
Resembles the classical elimination theory for polynomials.

Elimination process and splittings

A **splitting** of G is a representation of G as the fundamental groups of a graph of groups.
A splitting is **cyclic (abelian)** if all the edge groups are cyclic (abelian).

Elementary splittings:

$G = A *_C B, \quad G = A*_C = \langle A, t \mid t^{-1}Ct = C' \rangle,$

Free splittings:

$G = A * B$

Elimination processes and free actions

Infinite branches of an elimination process correspond precisely to the standard types of free actions:
linear case $\Longleftrightarrow$ **thin (or Levitt)** type
the quadratic case $\Longleftrightarrow$ **surface type (or interval exchange)**,
periodic structures $\Longleftrightarrow$ **toral (or axial)** type.

Bestvina-Feighn's elimination process

A powerful variation of the Makanin-Razborov's process for $\mathbb{R}$-actions.
Can be viewed as an asymptotic (limit) version of MR process.
Much simpler in applications but not algorithmic.

KM elimination process for $\mathbb{Z}^n$ actions

To solve equations in fully residually free groups we designed a variation of the elimination process for $\mathbb{Z}^n$ actions.

It **effectively** describes solution sets of finite systems of equations in $\mathbb{Z}^n$-groups in terms of **Triangular quasi-quadratic systems** (as in the case of fully residually free groups).

Non-standard version of Rip's machine

Kh., Myasnikov, and Serbin designed an elimination process for arbitrary non-Archimedean actions, i.e, free actions on Λ-trees.

This can be viewed as a **non-Archimedean (non-standard)** discrete, effective version of the original MR process.

Sketch of the proof of the theorem about Λ-free f.p. groups

Let G have a regular free length function in Λ.

Fix an embedding of G into $CDR(\Lambda, X)$ and construct a cancellation tree for each relation of G.

Sketch of the proof

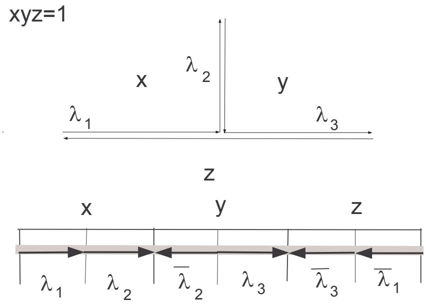

Figure: From the cancellation tree for the relation $xyz = 1$ to the generalized equation $(x = \lambda_1 \circ \lambda_2,\ y = \lambda_2^{-1} \circ \lambda_3,\ z = \lambda_3^{-1} \circ \lambda_1^{-1})$.

Sketch of the proof

Infinite branches of an elimination process correspond to abelian splittings of G:

linear case $\Longleftrightarrow$ **splitting as a free product.**

the quadratic case $\Longleftrightarrow$ **QH-subgroup**,

periodic structures $\Longleftrightarrow$ **abelian** vertex group or splitting as an HNN with abelian edge group.

After obtaining a splitting we apply EP to the vertex groups. We build the Delzant-Potyagailo hierarchy.

Sketch of the proof

A family $\mathcal{C}$ of subgroups of a torsion-free group G is called *elementary* if
(a) $\mathcal{C}$ is closed under taking subgroups and conjugation,
(b) every $C \in \mathcal{C}$ is contained in a maximal subgroup $\overline{C} \in \mathcal{C}$,
(c) every $C \in \mathcal{C}$ is small (does not contain F_2 as a subgroup),
(d) all maximal subgroups from $\mathcal{C}$ are malnormal.

G admits a *hierarchy* over $\mathcal{C}$ if the process of decomposing G into an amalgamated product or an HNN-extension over a subgroup from $\mathcal{C}$, then decomposing factors of G into amalgamated products and/or HNN-extensions over a subgroup from $\mathcal{C}$ etc. eventually stops.

Theorem (Delzant - Potyagailo (2001)). If G is a finitely presented group without 2-torsion and $\mathcal{C}$ is a family of elementary subgroups of G then G admits a hierarchy over $\mathcal{C}$.

Corollary. If G is a f.p. Λ-free group then G admits a hierarchy

Hyperbolic length functions Let G be a group, Λ an ordereg abelian group. A function $l : G \to \Lambda$ is called a δ-hyperbolic length function on G if

(L1) $\forall\, g \in G: \; l(g) \geqslant 0$ and $l(1) = 0$,

(L2) $\forall\, g \in G: \; l(g) = l(g^{-1})$,

(L3) $\forall\, g, h \in G: \; l(gh) \leq l(g) + l(h)$,

(L4) $\forall\, f, g, h \in G: \; c(f, g) \geq min\{c(f, h), c(g, h)\} - \delta$, where $c(f, g)$ is the Gromov's product:

$$c(g, f) = \frac{1}{2}(l(g) + l(f) - l(g^{-1}f)).$$

A δ-hyperbolic length function is called complete if $\forall g \in G$, and $\alpha \leq l(g)$ there is $u \in G$ such that $g = u \circ g_1$, where $l(u) = \alpha$.

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Free hyperbolic length functions A δ -hyperbolic length function is called free (δ-free) if
$\forall g \in G : \; g \neq 1 \rightarrow l(g^2) > l(g) \; (resp., l(g^2) > l(g) - c(\delta))$.

Problem

Find the structure of f.g. groups with δ-hyperbolic, δ-regular, δ-free length function, in $\mathbb{Z}^n$, where $l(\delta)$ is in the smallest component of $\mathbb{Z}^n$.

A.P. Grecianu (McGill) obtained first results in this direction.