Conjugacy growth series

and

languages in groups

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Growth functions, growth series, and languages

$X =$ finite set $\qquad X^* = \{$ words over $X\}$ $\qquad$ Language: $L \subseteq X^*$

Growth functions : $\mathbb{N} \to \mathbb{N}$

Strict: $str_L(i) := \#\{w \in L \mid l(w) = n\}$
Cumulative: $cum_L(i) := \#\{w \in L \mid l(w) \leq n\}$

Growth series (= generating functions) $\in \mathbb{C}[[z]]$:

Strict: $s_L(z) := \sum_{i=0}^{\infty} str_L(i) z^i$
Cumulative: $c_L(z) := \sum_{i=0}^{\infty} cum_L(i) z^i$

Rmk. $s_L(z) = (1 - z)c_L(z)$; i.e.,

$s_L(z)$ is rational iff $c_L(z)$ is rational.

(We'll use *strict* growth throughout.)

Fact. If L is a regular language (can be built from finite subsets of X^* using finitely many $\cup$, $\cap$, ^{c}, $\cdot$, * operations), then $s_L(z)$ is rational.

Spherical growth series and language for groups

G = group $\qquad$ X = finite inverse-closed generating set

d = path metric in Cayley graph $\qquad$ $|g| = d(\epsilon, g)$ for $g \in G$

$S(i) := \{g \in G \mid |g| = i\}$ $\qquad$ $B(i) := \{g \in G \mid |g| \leq i\}$

Spherical growth series $\sigma = \sigma_{G,X}(z) := \sum_{i=0}^{\infty} \#S(i) z^i$

(Ball) Growth series $\beta_{G,X}(z) := \sum_{i=0}^{\infty} \#B(i) z^i$

$<$ = total order on X $\qquad$ $<_{sl}$ = shortlex order on X^*

$\forall g \in G$, let $y_g :=$ shortlex least representative of g

Spherical language $\Sigma = \Sigma(G, X) = \{y_g \mid g \in G\}$

Fact. $\sigma_{G,X}(z) = (1 - z)\beta_{G,X}(z) = s_\Sigma(z)$.

Ex. $G = F_2 = \langle a, b \mid \ \rangle$, $X = \{a, b\}^{\pm 1}$.

$\Sigma = X^* \setminus (X^* \cdot \{aa^{-1}, a^{-1}a, bb^{-1}, b^{-1}b\} \cdot X^*)$, $\qquad$ $\sigma(z) = \frac{1+z}{1-3z}$

Ex. (ECHLPT) Σ is regular, hence σ is rational, for word hyperbolic, shortlex automatic groups

Spherical conjugacy growth series and language for groups

G = group $\quad X$ = finite inverse-closed generating set

$G/\sim_c$ = conjugacy classes; $\quad |\alpha|_c = \min\{|g| \mid g \in \alpha\}$ for $\alpha \in G/\sim_c$

$\widetilde{S}(i) := \{\alpha \in G \sim_c \mid |\alpha|_c = i\}$ $\quad \widetilde{B}(i) := \{\alpha \in G \sim_c \mid |\alpha|_c \leq i\}$

Spherical conjugacy growth series $\tilde{\sigma} = \tilde{\sigma}_{G,X}(z) := \sum_{i=0}^{\infty} \#\widetilde{S}(i) z^i$

(Ball) Conjugacy growth series $\tilde{\beta}_{G,X}(z) := \sum_{i=0}^{\infty} \#\widetilde{B}(i) z^i$

Consistent with Guba-Sapir, Rivin; differs from Margulis, others.

$<$ = total order on X $\quad <_{sl}$ = shortlex order on X^*

$\forall \alpha \in G/\sim_c$, let $z_\alpha :=$ shortlex least word rep. an elt. of G in α

Spherical conjugacy language $\widetilde{\Sigma} = \widetilde{\Sigma}(G, X) = \{z_\alpha \mid \alpha \in G/\sim_c\}$

Fact. $\tilde{\sigma}_{G,X}(z) = (1-z)\tilde{\beta}_{G,X}(z) = s_{\widetilde{\Sigma}}(z)$.

Rmk. Coornaert, Knieper: Asymptotic bounds on conjugacy growth functions using exponential cumulative rate of spherical growth, for torsion-free non-elementary word hyperbolic groups.

Ex. $G = S_3 = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1\rangle$, $X = \{a, b\}$.

$\Sigma = \{1, a, b, ab, ba, aba\}$, $\quad \sigma(z) = 1 + 2z + 2z^2 + z^3$

$\widetilde{\Sigma} = \{1, a, ab\}$, $\quad \tilde{\sigma}(z) = 1 + z + z^2$

Closure and graph products

Def. The **graph product** associated to a finite graph Λ with vertex groups G_i is the group
$G = \langle G_i \mid [g,h] = \lambda$ whenever $g \in G_i$, $h \in G_j$, and G_i, G_j adjacent$\rangle$.

Rmk. Graph has no edges $\Rightarrow G = *_i G_i$ is the free product.
All vertex pairs adjacent $\Rightarrow G = \times_i G_i$ is the direct product.

Thm. (Chiswell) Rationality of the spherical growth series σ is closed under graph product.

Thm. (H, Meier) Regularity of the spherical language Σ is closed under graph product.

Lem. (Rivin; CH) Rationality of the spherical conjugacy growth series $\tilde{\sigma}$ and regularity of the spherical conjugacy language $\widetilde{\Sigma}$ are closed under direct product.

(In fact, $\tilde{\sigma}_{G\times H, X\cup Y} = \tilde{\sigma}_{G,X}\tilde{\sigma}_{H,Y}$ and $\widetilde{\Sigma}(G\times H, X\cup Y) = \widetilde{\Sigma}(G,X)\widetilde{\Sigma}(H,Y)$.)

But what about free products and spherical conjugacy?...

Spherical conjugacy and free products

Ex. $\mathbb{Z} = \langle a \mid \ \rangle$, $X = \{a\}^{\pm 1}$: $\tilde{\sigma}(z) = \sigma(z) = \frac{1+z}{1-z}$.

Thm. (Rivin) The spherical conjugacy growth series $\tilde{\sigma}(F_2, X)$ for the free group $F_2 = \langle a, b \mid \ \rangle$ with respect to $X = \{a, b\}^{\pm 1}$ is not rational.

Thm. (Ciobanu, H) The spherical conjugacy language $\widetilde{\Sigma}(F_2, X)$ is not context-free.

Reason: WLOG $a < b$.

(Regular language) $\cap$ (context-free (CF) language) is CF.

$\widetilde{\Sigma}$ is CF $\Leftrightarrow$ $\widetilde{\Sigma} \cap (a^+ b^+ a^+ b^+)$ is CF.

$a^i b^j a^k b^l \in \widetilde{\Sigma}(F_2, X) \Leftrightarrow$ either $i > k$ or $i = k$ and $j \leq l$.

Latter is not CF - violates CF Pumping Lemma.

Dependence on generating set

Thm. (Stoll) Rationality of the spherical growth series depends upon the generating set.

Thm. (Hull, Osin) There is a finitely generated group G with finite index subgroup H such that the cumulative growth function $cum(i)$ associated to the spherical conjugacy language for G grows exponentially, but H has only two conjugacy classes.

Conj. Rationality of the spherical conjugacy growth series depends upon the generating set.

Q. Is the spherical conjugacy growth series $\tilde{\sigma}_{F_2,Y}$ rational for another finite inverse-closed generating set Y of F_2?

Free products of finite groups

Thm. (Ciobanu, H) Let G_i be a finite group and $X_i := G_i \setminus \epsilon_{G_i}$.
(1) The spherical conjugacy growth series $\tilde{\sigma}_{G_1 * G_2, X_1 \cup X_2}$ is rational iff $G_1 = G_2 = \mathbb{Z}/2\mathbb{Z}$.

(2) Let $<$ be an ordering on $X := X_1 \cup X_2$ satisfying $a < b$ for all $a \in G_1$, $b \in G_2$. The spherical conjugacy language $\widetilde{\Sigma}_{G_1 * G_2, X}$ is regular iff $G_1 = G_2 = \mathbb{Z}/2\mathbb{Z}$.

Reason: $z \frac{d}{dz} \tilde{\sigma}_{G_1 * G_2, X} =$

$|\widetilde{\Sigma}(G_1, X_1) \cup \widetilde{\Sigma}(G_2, X_2) \setminus \{\lambda\}| z + 2 \sum_{r=1}^{\infty} \sum_{e|r} \phi(e) (|X_1||X_2|)^{r/e} z^{2r} =$

$|\widetilde{\Sigma}(G_1, X_1) \cup \widetilde{\Sigma}(G_2, X_2) \setminus \{\lambda\}| z + 2 \sum_{d=1}^{\infty} \phi(d) \frac{|X_1||X_2| z^{2d}}{1 - |X_1||X_2| z^{2d}}$.

When $|X_1||X_2| > 1$, have analytic function on unit disk except at infinitely many poles.

Rmk. Thm above gives evidence for:

Conj. (Rivin) A word hyperbolic group G has rational spherical conjugacy growth series if and only if G is virtually cyclic.

Idea: The *spherical* conjugacy growth series is rarely rational.

Geodesic growth series and language for groups

$G =$ group $\quad X =$ finite inverse-closed generating set
$d =$ path metric in Cayley graph $\quad |w| = d(\epsilon, w)$ for $w \in X^*$

Geodesic language $\Gamma = \Gamma(G,X) = \{w \in X^* \mid l(w) = |w|\}$

Geodesic growth series $\gamma = \gamma_{G,X}(z) := s_\Gamma(z)$

$G/\sim_c = \{\text{conj.}\}$ classes $\quad |w|_c = \min\{|g| \mid g \in [w]_c\}$ for $w \in X^*$

Geodesic conjugacy language

$\tilde{\Gamma} = \tilde{\Gamma}(G,X) = \{w \in X^* \mid l(w) = |w|_c\}$

Geodesic conjugacy growth series $\tilde{\gamma} = \tilde{\gamma}_{G,X}(z) := s_{\tilde{\Gamma}}(z)$

Ex. $G = S_3 = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1\rangle$, $X = \{a, b\}$.

$\Sigma = \{1, a, b, ab, ba, aba\}$, $\quad \sigma(z) = 1 + 2z + 2z^2 + z^3$

$\widetilde{\Sigma} = \{1, a, ab\}$, $\quad \tilde{\sigma}(z) = 1 + z + z^2$

$\Gamma = \{1, a, b, ab, ba, aba, bab\}$, $\quad \gamma(z) = 1 + 2z + 2z^2 + 2z^3$

$\tilde{\Gamma} = \{1, a, b, ab, ba\}$, $\quad \tilde{\gamma}(z) = 1 + 2z + 2z^2$

Ex. Γ is regular, hence γ is rational, for: Word hyperbolic (Cannon), virtually abelian, geometrically finite hyperbolic (Neumann-Shapiro), Coxeter (Howlett), Garside (Charney-Meier) groups. Regularity of Γ can depend upon the generating set (Cannon).

Geodesics, conjugacy geodesics, and graph products

G = graph product of finite graph Λ with vertex groups $G_i = \langle X_i \rangle$ with each X_i finite, inverse-closed.

$X := \cup_i X_i$.

For each i, define a homomorphism $\pi_i : X^* \to (X_i \cup \{\$\})^*$ by

$$\pi_i(a) := \begin{cases} a & \text{if } a \in X_i \\ \$ & \text{if } a \in X_j \text{ and } X_i, X_j \text{ not adjacent} \\ 1 & \text{if } a \in X_j \text{ and } X_i, X_j \text{ adjacent} \end{cases}$$

Thm. (Ciobanu, H)

(1) Geodesic words for G over X satisfy

$$\Gamma(G,X) = \cap_i \; \pi_i^{-1}(\; \Gamma(G_i,X_i)(\$ \; \Gamma(G_i,X_i))^* \;).$$

(2) Conjugacy geodesic words for G over X satisfy

$$\tilde{\Gamma}(G,X) = \cap_i \; \pi_i^{-1}(\; \tilde{\Gamma}(G_i,X_i) \cup \tilde{U}_i \;) \quad \text{where}$$

$\tilde{U}_i := \{u_0\$u_1 \cdots \$u_m \mid m \geq 1 \text{ and } u_1, ..., u_{m-1}, u_m u_0 \in \tilde{\Gamma}(G_i, X_i)\}$.

The free group on 2 generators

Ex. $G = F_2 = \langle a, b \mid \ \rangle$, $X = \{a, b\}^{\pm 1}$. $\Lambda :\ \langle a \mid \rangle \bullet\!\!-\!\!-\!\!-\!\!\bullet \langle b \mid \rangle$

$\Gamma = \Sigma = X^* \setminus (X^* \cdot \{aa^{-1}, a^{-1}a, bb^{-1}, b^{-1}b\} \cdot X^*)$ are regular

$\Gamma(\langle a \mid \ \rangle, \{a^{\pm 1}\}) = a^* \cup a^{-1*}$ $\Gamma(\langle b \mid \ \rangle, \{b^{\pm 1}\}) = b^* \cup b^{-1*}$

$$\pi_1(a) = a, \quad \pi_1(a^{-1}) = a^{-1}, \quad \pi_1(b) = \$, \quad \pi_1(b^{-1}) = \$$$

$$\pi_2(a) = \$, \quad \pi_2(a^{-1}) = \$, \quad \pi_2(b) = b, \quad \pi_2(b^{-1}) = b^{-1}$$

$\widetilde{\Gamma} = \{w \in X^* \mid \pi_1(w) \in ((a^* \cup a^{-1*})(\$\ (a^* \cup a^{-1*}))^*)$ and

$\pi_2(w) \in ((b^* \cup b^{-1*})(\$\ (b^* \cup b^{-1*}))^*)\}$ is regular

$\gamma(z) = \sigma(z) = \frac{1+z}{1-3z}$ $\widetilde{\gamma}(z) = \frac{1+z-z^2-9z^3}{(1-3z)(1-z)(1+z)}$

$\widetilde{\Sigma}$ is not CF. $\widetilde{\sigma}(z)$ not rational.

Geodesics, conjugacy geodesics, and graph products, II

Cor. (Ciobanu, H) Regularity for the pair $\Gamma, \widetilde{\Gamma}$ of geodesic and geodesic conjugacy languages is closed under graph product.

Cor. (Ciobanu, H) Every right-angled Artin group, right-angled Coxeter group, and graph product of finite groups has a regular geodesic conjugacy language $\widetilde{\Gamma}$ and a rational geodesic conjugacy growth series $\tilde{\gamma}$.

Rmk. This gives new proof of:

Cor. (Loeffler, Meier, Worthington) Regularity of the geodesic language Γ is closed under graph product.

Geodesics, conjugacy geodesics, and graph products, III

Thm. (Ciobanu, H)

(1) Geodesic words: $\Gamma(G,X) = \cap_i \ \pi_i^{-1}(\ \Gamma(G_i,X_i)(\$\ \Gamma(G_i,X_i))^*\ )$.

(2) Conjugacy geodesic words: $\tilde{\Gamma}(G,X) = \cap_i \ \pi_i^{-1}(\ \tilde{\Gamma}(G_i,X_i) \cup \tilde{U}_i\ )$

$\tilde{U}_i := \{u_0\$u_1 \cdots \$u_m \mid m \geq 1$ and $u_1, ..., u_{m-1}, u_m u_0 \in \tilde{\Gamma}(G_i,X_i)\}$.

Notes on proof

• Using an extended generating set $T \supseteq X$, find a weightlex complete rewriting system for G.

$T := \{a_1 \cdots a_m \mid a_i \in X_{k_i}$ and $i \neq j \Rightarrow X_{k_i}, X_{k_j}$ adjacent$\}$.

• Have canonical homomorphism $T^* \to X^*$. Images of irreducible words = geodesic set $\mathcal{N}$ of normal forms over X.

• $w \in X^*$ is geodesic iff normal form for w can be obtained from w by operations of the form

Local exchange: $yuz \to ywz$ with $y,z \in X^*$, $u,w \in X_i^*$, $u =_{G_i} w$, and $l(u) = l(w)$.

Shuffle: $yuwz \to ywuz$ with $y,z \in X^*$, $u \in X_i^*$, $w \in X_j^*$, and i,j adjacent in Λ.

Geodesic series for direct and free products

Let G and H be groups with finite inverse-closed generating sets A and B, respectively. Let $X := A \cup B$.

Free products

Thm. (1) (Loeffler, Meier, Worthington)

$$\gamma_{G*H,X} = \frac{\gamma_{G,A}\gamma_{H,B}}{1-(\gamma_{G,A}-1)(\gamma_{H,B}-1)}.$$

(2) (Ciobanu, H)

$$\tilde{\gamma}_{G*H,X} - 1 = (\tilde{\gamma}_{G,A} - 1) + (\tilde{\gamma}_{H,B} - 1) - z\frac{d}{dz}\ln\left[1 - (\gamma_{G,A} - 1)(\gamma_{H,B} - 1)\right].$$

Direct products

Thm. (1) (Loeffler, Meier, Worthington) $\gamma_{G\times H,X} = \sum_{i=0}^{\infty} c_i z^i$

where $\gamma_{G,A}(z) = \sum_{i=0}^{\infty} a_i z^i$, $\tilde{\gamma}_{H,B}(z) = \sum_{i=0}^{\infty} b_i z^i$, and $c_i := \sum_{j=0}^{i} \binom{i}{j} a_j b_{i-j}$.

(2) (Ciobanu, H) $\tilde{\gamma}_{G\times H,X} = \sum_{i=0}^{\infty} t_i z^i$

where $\tilde{\gamma}_{G,A}(z) = \sum_{i=0}^{\infty} r_i z^i$, $\tilde{\gamma}_{H,B}(z) = \sum_{i=0}^{\infty} s_i z^i$, and $t_i := \sum_{j=0}^{i} \binom{i}{j} r_j s_{i-j}$.

Geodesic series for direct and free products, cont'd

Ex. *Right-angled Coxeter group:*

$G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^2 = 1 \rangle \qquad X = \{a, b, c\}$

The spherical language $\Sigma(G, X)$ and spherical conjugacy language $\widetilde{\Sigma}(G, X)$ are regular, and the spherical and spherical conjugacy series $\sigma_{G,X}$ and $\tilde{\sigma}_{G,X}$ are rational.

The geodesic language

$$\Gamma(G, X) = \{\lambda, c\}(bc)^*\{\lambda, a\}(bc)^*\{\lambda, b\} \cup \{\lambda, b\}(cb)^*\{\lambda, a\}(cb)^*\{\lambda, c\}$$

and the geodesic conjugacy language

$$\widetilde{\Gamma}(G, x) = \{b, c, ab, ac, ba, ca\} \cup (bc)^*\{\lambda, a, bac\}(bc)^* \cup (cb)^*\{\lambda, a, cab\}(cb)^*$$

are regular, and the corresponding rational growth series are

$$\gamma_{G,X}(z) = \frac{1+z+z^2-z^3}{(1-z)^2} \qquad \tilde{\gamma}_{G,X} = \frac{1+3z+4z^2-9z^4+z^5+4z^6}{(1-z)^2(1+z)^2}$$

Amalgamated products of finite groups

Ex. *Free product of finite groups:*

G, H finite groups $\quad A := G \setminus \{1_G\},\ B := H \setminus \{1_H\}$

Let $m := |A| = |G| - 1$ and $n := |B| = |H| - 1$.

$\tilde{\Gamma}(G, A) = \Gamma(G, A) = A \cup \{\lambda\} \quad \tilde{\Gamma}(H, B) = \Gamma(H, B) = B \cup \{\lambda\}$

$\tilde{\gamma}_{G,A} = \gamma_{G,A} = mz + 1 \quad \tilde{\gamma}_{H,B} = \gamma_{H,B} = nz + 1.$

$\tilde{\gamma}(G * H, A \cup B)(z) = \frac{1+(m+n)z+mnz^2-mn(m+n)z^3}{1-mnz^2}.$

Ex. For $P := PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b, c \mid a^2 = 1, b^2 = c, bc = 1\rangle$ with $X = \{a, b, c\}$, the geodesic conjugacy growth series is

$\tilde{\gamma}_{P,X}(z) = \frac{1+3z+2z^2-6z^3}{1-2z^2}.$

Thm. (Ciobanu, H) If G and H are finite groups with a common subgroup K, then the free product $G *_K H$ of G and H amalgamated over K, with respect to the generating set $X := G \cup H \cup K - \{1\}$, has regular geodesic conjugacy language $\tilde{\Gamma}(G *_K H, X)$ and rational geodesic conjugacy series $\tilde{\gamma}_{G *_K H, X}$.

An in-between growth function

Recall $y_g :=$ shortlex least rep. of $g \in G$.

Equality conjugacy language $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}(G,X) := \{y_g \mid |g| = |g|_c\}$

Equality conjugacy growth series $\tilde{\epsilon} = \tilde{\epsilon}_{G,X} := s_{\widetilde{\mathcal{E}}}(z)$

$$\begin{array}{ccccc} \widetilde{\Sigma} & \subseteq & \widetilde{\mathcal{E}} & \subseteq & \widetilde{\Gamma} \\ \cup\!| & & & & \cup\!| \\ \Sigma & & \subseteq & & \Gamma \end{array}$$

Lem. (Rivin) If $G = F_2$ and $X =$ (free basis)$^{\pm 1}$ then $\tilde{\epsilon}$ is rational.

(In fact $\widetilde{\mathcal{E}} = \widetilde{\Gamma}$ is regular and $\tilde{\epsilon} = \tilde{\gamma}$.)

Q. Is regularity of the equality conjugacy language preserved under graph product?

Q. Is rationality of the equality conjugacy growth series preserved by direct and free products?

More open questions

(1) Let $F = F(a, b)$ be the free group on generators a and b. Note that the spherical conjugacy language $\widetilde{\Sigma}(F, \{a^{\pm 1}, b^{\pm 1}\})$ is computable. Is this set an indexed or context sensitive language?

(2) Let G be a word hyperbolic group and let X be a finite inverse-closed generating set for G. Is the geodesic conjugacy language $\widetilde{\Gamma}(G, X)$ necessarily regular?

(3) Many groups G (e.g. closed surface groups (Cannon)) have a generating set for which the rational spherical growth series σ satisfies $\sigma(1) = 1/\chi(G)$, where χ is the Euler characteristic of G. What connections, if any, are there between spherical/geodesic conjugacy growth series and homological properties of G?

(4) The **complete growth series** for G over X is given by $\beta(z) := \sum_{i=0}^{\infty}(\sum_{|g|=i} g)z^i$ in $\mathbb{Z}[G][[z]]$. **Thm.** (Grigorchuk, Nagnibeda) For word hyperbolic groups β is "rational".

Is the series $\tilde{\beta}(z) := \sum_{i=0}^{\infty}(\sum_{|g|=|g|_c=i} g)z^i \in \mathbb{Z}[G][[z]]$ rational for word hyperbolic groups?