Limits of hyperbolic groups are $\mathbb{Z}^n$-hyperbolic

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The talk is based on the results of two preprints:

[GKMS]: “Groups acting on $\Lambda$-metric spaces” by Andrei-Paul Grecianu, Alexei Kvaschuk, Alexei Myasnikov, Denis Serbin, 2013.

[MS]: “Hyperbolic length functions on limits of torsion-free hyperbolic groups” by Alexei Myasnikov and Denis Serbin, 2013.
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This order on $\mathbb{Z}^n$ is non-archimedean, for example

$t \cdot (a_1, \ldots, a_{n-1}, 0) < (0, \ldots, 0, 1)$

for any $t \in \mathbb{N}$ and $a_1, \ldots, a_{n-1} \in \mathbb{Z}$. 

Limits of hyperbolic groups are $\mathbb{Z}^n$-hyperbolic
A $\Lambda$-metric space is a non-empty set $X$ equipped with a mapping $d : X \times X \rightarrow \Lambda$ which satisfies the usual metric axioms with $\mathbb{R}$ replaced by $\Lambda$:

(i) $\forall x, y \in X : d(x, y) \geq 0$

(ii) $\forall x, y \in X : d(x, y) = 0 \iff x = y$

(iii) $\forall x, y \in X : d(x, y) = d(y, x)$

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- \( \Lambda/\Lambda_0 \) is an ordered abelian group since \( \Lambda_0 \) is convex,
- the metric \( d_Y \) on \( Y \) is defined by \( d_Y([x], [y]) = d(x, y) + \Lambda_0 \).
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Let \(\delta \in \Lambda\) with \(\delta \geq 0\). Then \((X, d)\) is \(\delta\)-hyperbolic with respect to \(v\) if, for all \(x, y, z \in X\),

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**Fact.** If $(X, d)$ is $\delta$-hyperbolic with respect to $v$, and $t$ is any other point of $X$, then $(X, d)$ is $2\delta$-hyperbolic with respect to $t$. 

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  $X$ is $\delta'$-hyperbolic for $\delta' = (0, 1) \in \mathbb{Z}^2 - \mathbb{Z}$.  

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**Theorem [GKMS].** If $X$ is geodesic then every minimal isometry of $X$ is either elliptic, or parabolic, or hyperbolic, or an inversion.
We say that a group $G$ acts freely (without inversions) on a $\delta$-hyperbolic $\Lambda$-metric space $(X, d)$ if every $g \in G$ acts on $X$ as a hyperbolic isometry.

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  - Do groups discriminated by $G$ act freely on $\delta$-hyperbolic $\mathbb{Z}^n$-metric spaces?
Theorem [Kharlampovich, Myasnikov, 2010]. Every finitely generated group discriminated by a torsion-free word-hyperbolic group $G$ embeds into the Lyndon’s completion $G^\mathbb{Z}[t]$ of $G$. 
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Lyndon’s completion $G^{\mathbb{Z}[t]}$ of a group $G$ is the union of the infinite chain of groups

$$G = G_0 < G_1 < \cdots < G_n < \cdots$$

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Here, to construct $G_{n+1}$ from $G_n$ let $\{\langle u_i \rangle \mid i \in I\}$ be the set of representatives of conjugacy classes of proper cyclic centralizers in $G_n$. Then

$$G_{n+1} = \langle G_n, s_{i,j} \ (i \in I, j \in \mathbb{N}) \mid [s_{i,j}, u_i] = [s_{i,j}, s_{i,k}] = 1, (i \in I, j, k \in \mathbb{N}) \rangle$$
Given a torsion-free word-hyperbolic group $G$ and its Cayley graph $\Gamma$ with respect to a finite generating set, we would like to construct a $\delta$-hyperbolic $\mathbb{Z}[t]$-metric space $\Gamma^{\mathbb{Z}[t]}$ which $G^{\mathbb{Z}[t]}$ (as well as all its subgroups) acts freely upon.
Scheme of the proof

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Denote \( \Gamma_0 = \Gamma \). Then \( \Gamma_1 \) is a \( \delta \)-hyperbolic \( \mathbb{Z}[t] \)-metric space obtained as the union of the chain

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\Gamma_0 = \Gamma_0^\mathbb{Z} \subset \Gamma_0^{\mathbb{Z}^2} \subset \cdots \subset \Gamma_0^{\mathbb{Z}^n} \subset \cdots
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where \( \Gamma_0^{\mathbb{Z}^n} \) is obtained from \( \Gamma_0^{\mathbb{Z}^{n-1}} \) by the construction described below. The group \( G_1 \) acts freely on \( \Gamma_1 \).
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Similarly we construct \( \Gamma_2 \) starting from \( \Gamma_1 \) so that \( G_2 \) acts freely on \( \Gamma_2 \) etc.
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Similarly we construct $\Gamma_2$ starting from $\Gamma_1$ so that $G_2$ acts freely on $\Gamma_2$ etc. Eventually,

$$\Gamma^{\mathbb{Z}[t]} = \bigcup_{n \in \mathbb{N}} \Gamma_n$$

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Let $\mathcal{C} = \{C_i \mid i \in I\}$ be a set of representatives of conjugacy classes of proper cyclic centralizers in $G$, that is, every proper cyclic centralizer in $G$ is conjugate to one from $\mathcal{C}$, and no two centralizers from $\mathcal{C}$ are conjugate. Denote by $U$ the set of generators of centralizers from $\mathcal{C}$.
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For any $u \in U$ there exist $u_-, u_+ \in \partial \Gamma_0$ fixed by $u$ and denote by $\partial U$ the set

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Informally, $\Gamma_0^{\mathbb{Z}^2}$ is a $\mathbb{Z}$-tree

- whose vertices are copies of $\Gamma_0$, [Limits of hyperbolic groups are $\mathbb{Z}^n$-hyperbolic]
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Informally, \( \Gamma_0^{\mathbb{Z}^2} \) is a \( \mathbb{Z} \)-tree

- whose vertices are copies of \( \Gamma_0 \),
- each edge \((v_1, v_2)\) corresponds to a pair of ends \((a, b) \in \partial U \times \partial U \subset \partial \Gamma_0(v_1) \times \partial \Gamma_0(v_2)\).
Limits of hyperbolic groups are $\mathbb{Z}^n$-hyperbolic.
Theorem [MS]. $\Gamma_0^{\mathbb{Z}^2}$ is a geodesic $\mathbb{Z}^2$-metric space which is $\delta$-hyperbolic with respect to $1_\varepsilon$. 
\( \Gamma_0^{\mathbb{Z}^2} \) as a geodesic \( \mathbb{Z}^2 \)-metric space

**Theorem [MS].** \( \Gamma_0^{\mathbb{Z}^2} \) is a geodesic \( \mathbb{Z}^2 \)-metric space which is \( \delta \)-hyperbolic with respect to \( 1_{\varepsilon} \).

For every point \( x \in \Gamma_0^{\mathbb{Z}^2} \) there exist (infinitely many) paths in \( \Gamma_0^{\mathbb{Z}^2} \) of the form

\[ [1_{\varepsilon}, a(e_1)] \cdot [b(e_1), a(e_2)] \cdots [b(e_n), a(e_n)] \cdot [b(e_{n+1}), x], \]

where \( a(e_i), b(e_i) \in \partial U \) and each piece is a quasi-geodesic in the corresponding copy of \( \Gamma_0 \). Denote the set of all such paths by \([x]\).
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The label of each path \( \gamma \in [x] \) is a \( \mathbb{Z}^2 \)-word of length \( d_2(1_\varepsilon, x) \) over the alphabet \( S \cup S^{-1} \). Since \( \Gamma_0^{\mathbb{Z}^2} \) is homogeneous (in terms of reading \( \mathbb{Z}^2 \)-words over \( S \cup S^{-1} \)), we can translate \( \gamma \) to any point of \( \Gamma_0^{\mathbb{Z}^2} \). Hence, we can concatenate paths and then “reduce” them.
Proposition [MS]. The set \{[x] \mid x \in \Gamma_0^{\mathbb{Z}^2}\} forms a group \(H\): the product \([x][y]\) is equal to \([z]\), where \(z \in \Gamma_0^{\mathbb{Z}^2}\) is obtained as the end-point of any \(\gamma \in [y]\) translated to \(x\).
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Lemma [MS]. \( H \) acts on \( \Gamma_0^{\mathbb{Z}^2} \): if \( h = [x] \in H \) and \( y \in \Gamma_0^{\mathbb{Z}^2} \) then \( h \cdot y = z \), where \([z] = [x][y] \). The action is isometric and free.
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Corollary [MS]. \(H\) is \(\mathbb{Z}^2\)-hyperbolic.
Isometries of $\Gamma_{0}^{\mathbb{Z}^{2}}$

**Proposition [MS].** The set $\{[x] \mid x \in \Gamma_{0}^{\mathbb{Z}^{2}}\}$ forms a group $H$: the product $[x][y]$ is equal to $[z]$, where $z \in \Gamma_{0}^{\mathbb{Z}^{2}}$ is obtained as the end-point of any $\gamma \in [y]$ translated to $x$.

**Lemma [MS].** $H$ acts on $\Gamma_{0}^{\mathbb{Z}^{2}}$: if $h = [x] \in H$ and $y \in \Gamma_{0}^{\mathbb{Z}^{2}}$ then $h \cdot y = z$, where $[z] = [x][y]$. The action is isometric and free.

**Corollary [MS].** $H$ is $\mathbb{Z}^{2}$-hyperbolic.

**Theorem [MS].** $H \cong \langle G, \hat{u} \mid \hat{u}, u \rangle = 1$ ($u \in U$)

In other words, $H$ is obtained from $G$ by extending all centralizers of $G$ by $\mathbb{Z}$. Limits of hyperbolic groups are $\mathbb{Z}^{n}$-hyperbolic.
Continuing by induction, we obtain the chain of $\delta$-hyperbolic $\mathbb{Z}^n$-metric spaces

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\Gamma_0 = \Gamma_0^0 \subset \Gamma_0^2 \subset \cdots \subset \Gamma_0^n \subset \cdots
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whose union we denote $\Gamma_1$. Similarly we “complete” $\Gamma_1$ to obtain $\Gamma_2$ etc. Eventually,

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Continuing by induction, we obtain the chain of $\delta$-hyperbolic $\mathbb{Z}^n$-metric spaces

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**Theorem [MS].** Any finitely generated group discriminated by $G$ (viewed as a subgroup of $G^\mathbb{Z}[t]$) is $\mathbb{Z}^n$-hyperbolic.
\[ G^{\mathbb{Z}[t]} \text{ acts freely on } \Gamma^{\mathbb{Z}[t]} \]

THANK YOU!