The Geometry of Rings

Chris Peikert
Georgia Institute of Technology

SCPQ
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LWE Over Rings (Over-Simplified) [LPR'10]

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$$
\|e + f\| \leq \|e\| + \|f\| \qquad \|e \cdot f\| \leq \sqrt{n} \cdot \|e\| \cdot \|f\|.
$$

"Expansion factor" $\sqrt{n}$ is worst-case. ("On average," $\approx \sqrt{\log n}$.)

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$$c_1 \leftarrow R_q \quad \text{and} \quad c_0 = -c_1 \cdot s + e \in R_q$$

  and output $c(S) = c_0 + c_1 S \in R_q[S]$. (Notice: $c(s) = e \bmod q$.)

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  Many mults $\Rightarrow$ large power of expansion factor $\Rightarrow$ tiny error rate $\alpha \Rightarrow$ big parameters!

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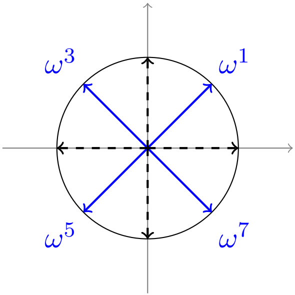
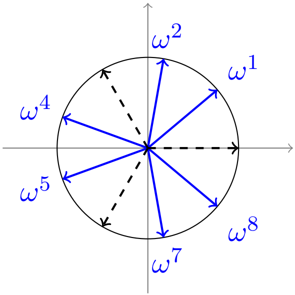


$\Phi_8(X) = 1 + X^4$



$\Phi_9(X) = 1 + X^3 + X^6$

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✗ Provable hardness also degrades with expansion factor: pay twice!

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   - ✔ No expansion factor anywhere
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Based on:

LPR'10 V. Lyubashevsky, C. Peikert, O. Regev.
"On Ideal Lattices and Learning with Errors Over Rings."

LPR'12 V. Lyubashevsky, C. Peikert, O. Regev.
"A Toolkit for Ring-LWE Cryptography."

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Key Facts

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- Bottom line: can reduce operations in R to independent operations in prime-power cyclotomic rings $\mathbb{Z}[X_i]/\Phi_{m_i}(X_i)$.

Canonical Geometry of R

- $R = \mathbb{Z}[X]/\Phi_m(X)$ has $n = \varphi(m)$ ring embeddings (homomorphisms) into $\mathbb{C}$, each given by evaluation at a root of Φ_m:

$$X \mapsto \omega^i \text{ for each } i \in \mathbb{Z}_m^*.$$

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✔ Expansion is element-specific. No more ring "expansion factor."

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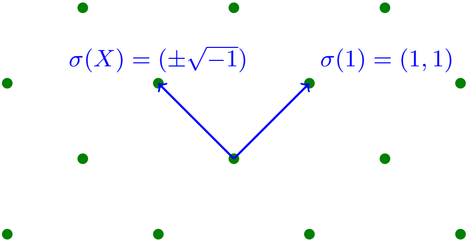

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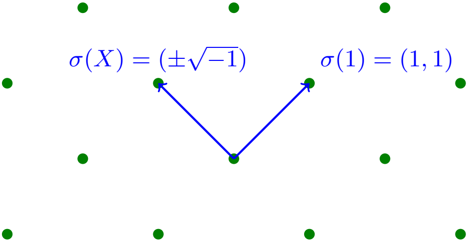

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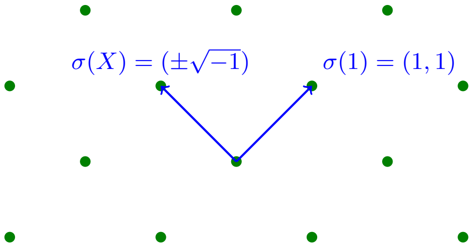

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So coefficient/canonical embeddings equivalent (up to $\sqrt{n}$ scaling).

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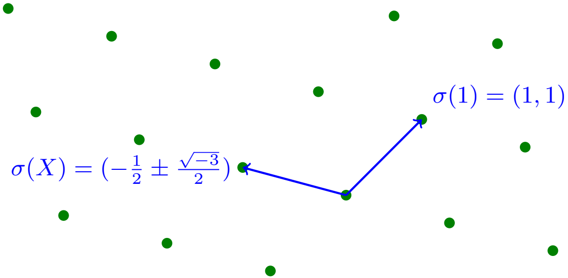

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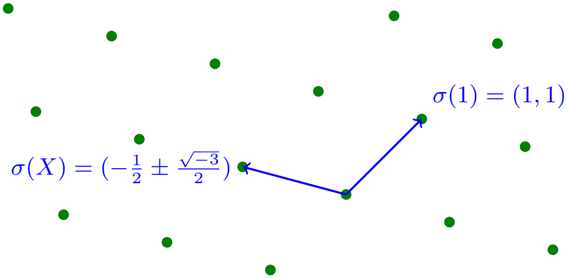

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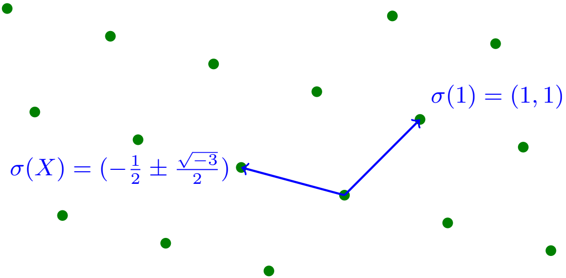

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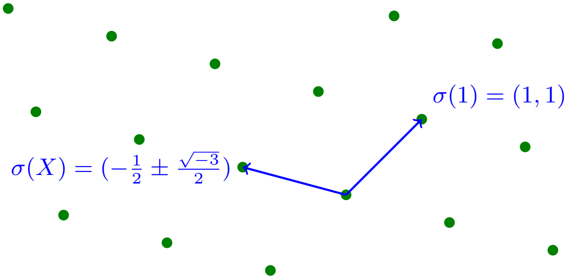

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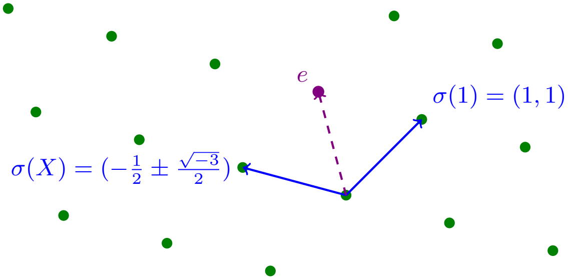

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  E.g., $\|e\| = \|1\| = \|X\| = \sqrt{n}$ <u>but</u> $e = 1 + X$.

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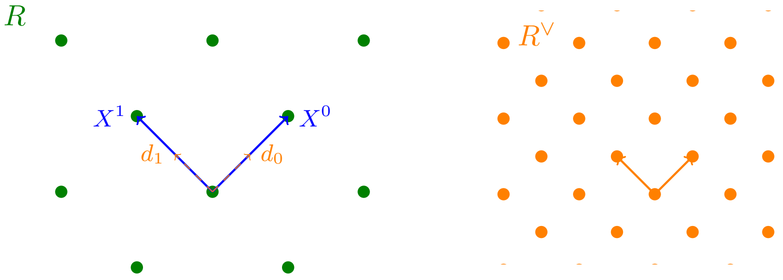

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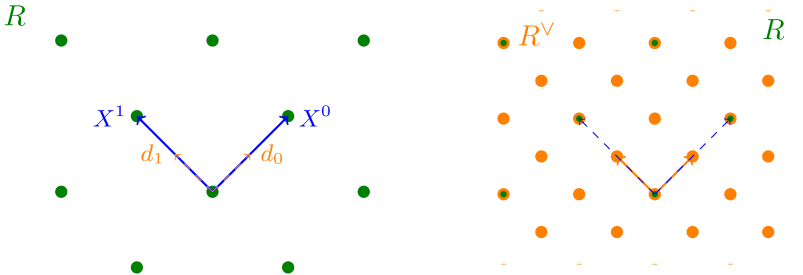

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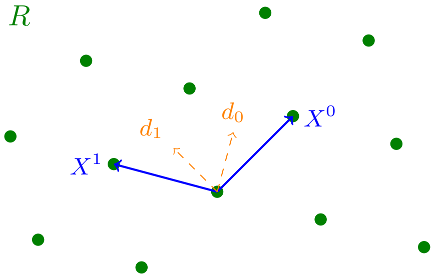

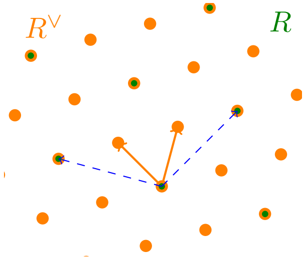

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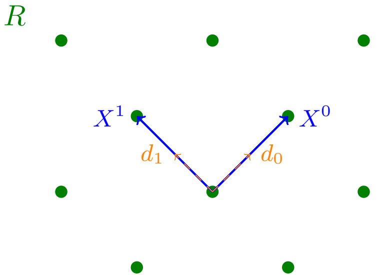

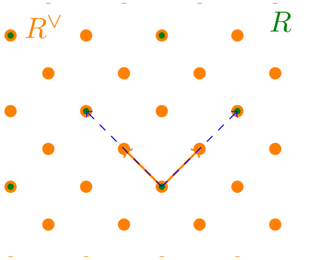

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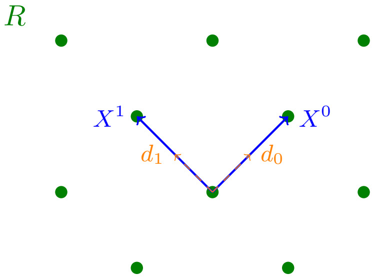

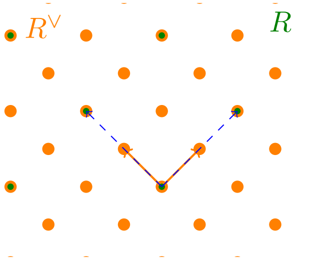

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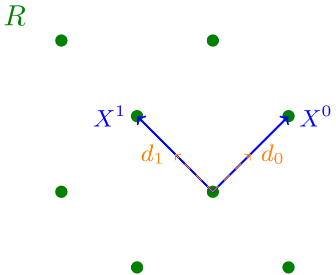

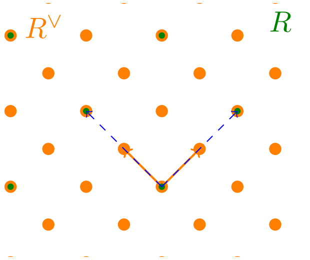

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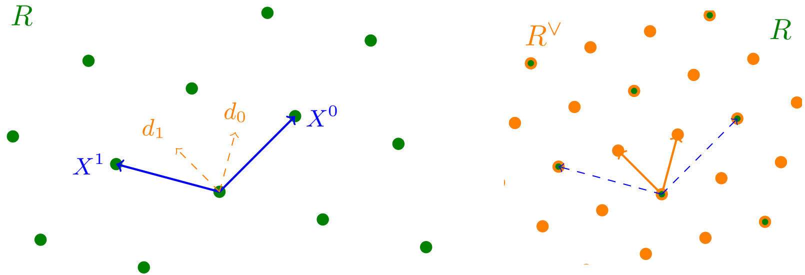

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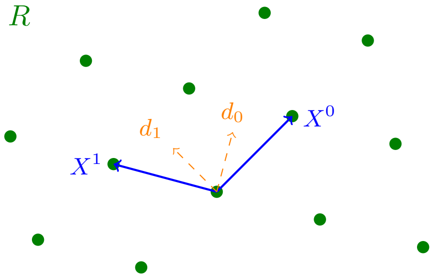

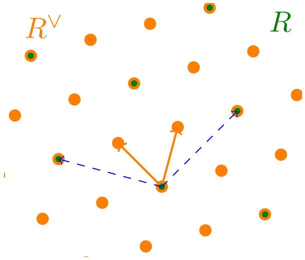

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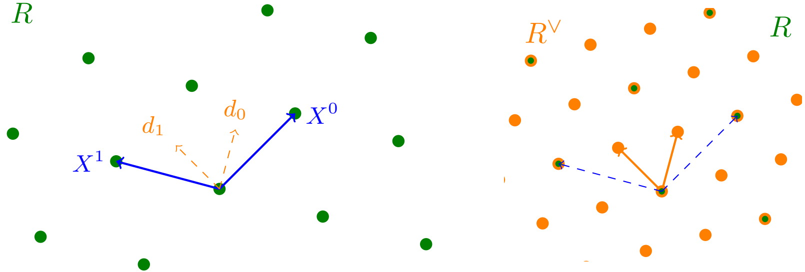

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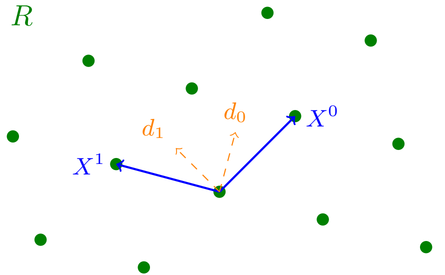

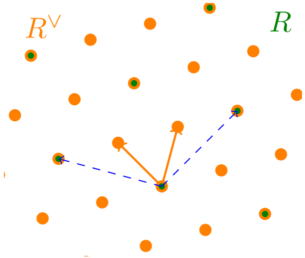

Ring-LWE: The Complete Definition [LPR'10]

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Theorem

For any m, ring-LWE with error std. dev. $\alpha q \geq 6^*$ is (quantumly) as hard as $\tilde{O}(n/\alpha)$-SVP on any ideal lattice in R.

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  and output $c(S) = c_0 + c_1 S \in R_q^\vee[S]$. (Note: $c(s) = e \bmod qR^\vee$.)
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  - ⋆ So need $q \approx 6^k m^{k-1} \sqrt{n} \approx (6m)^k$ to decrypt deg-k ciphertexts.
    Versus $q \approx \gamma^{k-1} n^k$ via expansion factor $\gamma \gg \sqrt{n}$.
    $\Rightarrow \approx \gamma^{k-1}$ factor improvement in error rate.

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