Polynomial time cryptanalysis of the Commutator Key Exchange Protocol

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Bar-Ilan University

Symbolic Computations and Post-Quantum Crypto Seminar
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In this lecture: Only passive adversaries.
The kernel on which more involved PKC is built.

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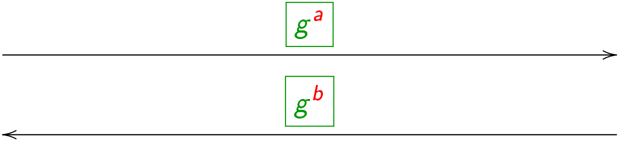

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$$\underbrace{g^x \mapsto x}_{\text{Discrete Logarithm Problem}} \geq \underbrace{(g^a, g^b) \mapsto g^{ab}}_{\text{Diffie–Hellman Problem (DHP)}}$$

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$10,000$ bits prime for "eternal" security? Impractical.

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How about noncommutative groups?

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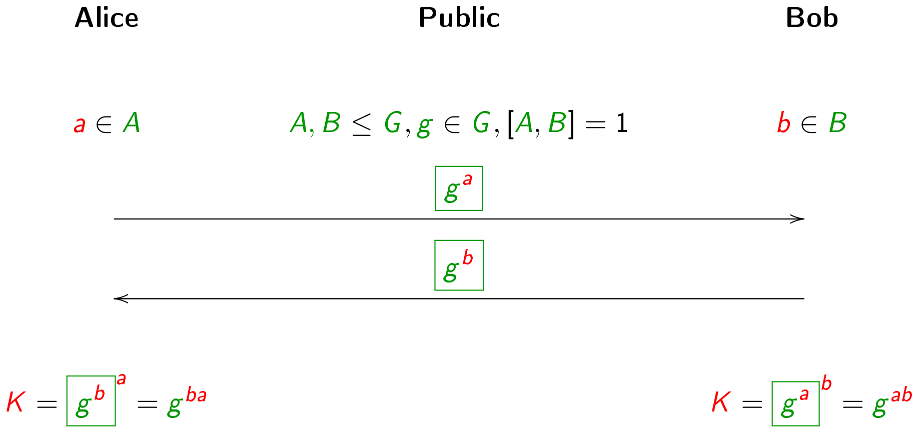
Ko–Lee–Cheon–Han–Kang–Park 2000. G noncommutative.

$g^x := x^{-1}gx$.



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Security parameters: $m := |i| + \ell$, N.

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$\therefore$ May work in the image of $\mathbf{B}_N$ in $\mathsf{GL}_{\binom{N}{2}}(\mathbb{Z}[t^{\pm 1}, \frac{1}{2}])$.

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$\therefore$ For prime $p \gtrsim 2^{N^2 m}$ and irreducible $f(t)$ of degree $\gtrsim 2m$,

$$(2^{2Nm} t^m) \cdot \mathrm{LK}(x) = (2^{2Nm} t^m) \cdot \mathrm{LK}(x) \bmod (p, f(t)) \in \mathrm{GL}_{\binom{N}{2}}(\mathbb{Z}[t]/\langle p, f(t) \rangle).$$

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$n := \binom{N}{2}$, roughly N^2. Henceforth, $G \leq \mathrm{GL}_n(\mathbb{F})$.

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Assume $G \cong^{\text{eff}}$ matrix group. Think G is a matrix group.

$$\boxed{g^a} = a^{-1} g a \iff a \cdot \boxed{g^a} = g \cdot a$$

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Second Braid Diffie–Hellman KEP

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Cha–Ko–Lee–Han–Cheon 2001.

Alice	**Public**	**Bob**
$a_1 \in A_1, a_2 \in A_2$	$A_1, A_2, B_1, B_2 \leq G, g \in G$	$b_1 \in B_1, b_2 \in B_2$
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Cheon–Jun 2003. Similar representation attack:

$$c = a_1 g a_2 \iff \boxed{a_1^{-1}} \cdot c = g \cdot a_2.$$

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$$\Pr(|\alpha_1 A_1 + \cdots + \alpha_m A_m| \neq 0) \geq 1 - \frac{n}{|\mathbb{F}|}.$$

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In our case, $|\mathbb{F}| > 2^n \gg n$.

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5. Complexity: $kn^6 + n^2 n^6 = n^8$.

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$a \in \langle a_1, \ldots, a_k \rangle, b \in \langle b_1, \ldots, b_k \rangle \leq G \leq \mathrm{GL}_n(\mathbb{F})$.

Commutator KEP Problem. $({b_1}^{a}, \ldots, {b_k}^{a}, {a_1}^{b}, \ldots, {a_k}^{b}) \mapsto a^{-1}b^{-1}ab$.

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3. $\exists$ solution: (a, b). Let $(\tilde{a}, \tilde{b})$ be one.

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The Centralizer KEP (Shpilrain–Ushakov 2006)

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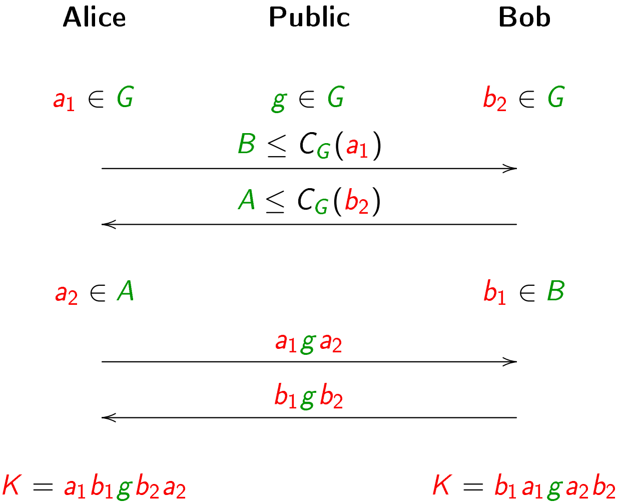

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$a_1 \in G$	$g \in G$	$b_2 \in G$
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	$\xrightarrow{a_1 g a_2}$	
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3. The other problems (CSP, Multiple CSP,...) on which braid-based PKC may be based.

The Triple Decomposition KEP (Kurt 2005)

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Alice **Public** **Bob**

a, a_1, a_2, x_1, x_2

$$\begin{tikzcd}[row sep=small]
A & A_1 \arrow[d, dash] & A_2 \arrow[d, dash] & X_1 \arrow[d, dash] & X_2 \arrow[d, dash] & & \\
& Y_1 & Y_2 & B_1 & B_2 & B & \leq G
\end{tikzcd}$$

y_1, y_2, b_1, b_2, b

$$\xrightarrow{\boxed{ax_1},\ \boxed{x_1^{-1}a_1x_2},\ \boxed{x_2^{-1}a_2}}$$

$$\xleftarrow{\boxed{b_1y_1},\ \boxed{y_1^{-1}b_2y_2},\ \boxed{y_2^{-1}b}}$$

$$a\boxed{b_1y_1}a_1\boxed{y_1^{-1}b_2y_2}a_2\boxed{y_2^{-1}b} = \underbrace{ab_1a_1b_2a_2b}_{K} = \boxed{ax_1}b_1\boxed{x_1^{-1}a_1x_2}b_2\boxed{x_2^{-1}a_2}b$$

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$$\begin{array}{cccccc} A & A_1 & A_2 & X_1 & X_2 & \\ & | & | & | & | & \\ & Y_1 & Y_2 & B_1 & B_2 & B \end{array} \leq G$$

y_1, y_2, b_1, b_2, b

$$\xrightarrow{\boxed{ax_1},\ \boxed{x_1^{-1}a_1x_2},\ \boxed{x_2^{-1}a_2}}$$

$$\xleftarrow{\boxed{b_1y_1},\ \boxed{y_1^{-1}b_2y_2},\ \boxed{y_2^{-1}b}}$$

$$a\boxed{b_1y_1}a_1\boxed{y_1^{-1}b_2y_2}a_2\boxed{y_2^{-1}b} = \underbrace{ab_1a_1b_2a_2b}_{K} = \boxed{ax_1}b_1\boxed{x_1^{-1}a_1x_2}b_2\boxed{x_2^{-1}a_2}b$$

THANK YOU!